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Indecomposable $U_q(s\ell_n)$ modules for $q^h = -1$ and BRS intertwiners

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Abstract
A class of indecomposable representations of $U_q(s\ell_n)$ is considered for q, an even root of unity ($q^h = -1$) exhibiting a similar structure as (height h) indecomposable lowest weight Kac–Moody modules associated with chiral conformal field theory. In particular, $U_q(s\ell_n)$ counterparts of the Bernard–Felder BRS operators are constructed for $n = 2, 3$. For $n = 2$ a pair of dual $d_2(h) = h$-dimensional $U_q(s\ell_2)$ modules gives rise to a $2h$-dimensional indecomposable representation including those studied earlier in the context of tensor-product expansions of irreducible representations. For $n = 3$ the interplay between the Poincaré–Birkhoff–Witt and (Lusztig) canonical bases is exploited in the study of $d_3(h) = \frac{h(h+1)(2h+1)}{6}$-dimensional indecomposable modules and of the corresponding intertwiners.

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1. Introduction

Quantum-group representations at q, a root of unity, arise in the study of chiral components of a Wess–Zumino–Novikov–Witten (WZNW) [1, 2] current algebra model [3–11]. Such chiral models necessarily involve an extended phase space with unphysical degrees of freedom. The Verma modules of a Kac–Moody current algebra (or, rather, their Wakimoto extensions [12]) were shown to give rise to a Becchi–Rouet–Stora (BRS) cohomology [13, 14]. A quantum group counterpart of this construction was worked out for $U_q(s\ell_2)$ in [7] where a complex of partially equivalent $U_q(s\ell_2)$ modules and intertwining 'BRS' maps was exhibited. Motivated by this paper we study dual pairs of finite-dimensional indecomposable representations of $U_q(s\ell_n)$ and the intertwining maps between them.

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Dual pairs of h-dimensional indecomposable $U_q(s\ell_2)$ modules first appeared in tensor-product expansions of irreducible ones as subrepresentations and subquotients of $2h$-dimensional indecomposable modules [15–18]. We introduce them from the outset as counterparts of indecomposable affine Kac–Moody modules of the chiral current algebra. The physical representations then appear as appropriate subquotients.

After some preliminary details about dual representations and linear and antilinear antiinvolutions in $U_q(s\ell_n)$ (section 2) we give in section 3 a comprehensive study of dual pairs of h-dimensional indecomposable $U_q(s\ell_2)$ modules for

$$q^h = -1 \qquad q + \overline{q} = 2\cos\frac{\pi}{h}. \tag{1.1}$$

Our attention is focused on two families of pairs, $(\mathcal{C}_p, \mathcal{C}_{2h-p})$ and $(\mathcal{V}_p, \mathcal{V}_{2h-p})$. The representations $\mathcal{C}_p$ and $\mathcal{C}_{2h-p}$ are cyclic, $\mathcal{V}_p$ has one highest weight (HW) and two lowest weight (LW) vectors for $0 < p < h$, while $\mathcal{V}_{2h-p}$ has one LW and two HW vectors (the modules $\mathcal{C}_h$ and $\mathcal{V}_h$ being equivalent and irreducible). For both pairs we define intertwining maps ('BRS operators') $Q^{h-p} : \mathcal{C}_p \to \mathcal{C}_{2h-p}$ ($\mathcal{V}_p \to \mathcal{V}_{2h-p}$) whose $(h-p)$-dimensional kernels $\mathcal{I}_{h-p}$ carry isomorphic irreducible representations (IR) of $U_q(s\ell_2)$. The 'physical' p-dimensional IR appears as a quotient $\mathcal{C}_p/\mathcal{I}_{h-p} \simeq \mathcal{V}_p/\mathcal{I}_{h-p}$. Both $\mathcal{C}_p$ and $\mathcal{V}_p$ admit a unique invariant bilinear form which gives rise to a non-degenerate inner product on the factor space (proposition 3.3).

In section 4 we construct $2h$-dimensional indecomposable representations $\mathcal{D}_p$ and $\mathcal{W}_p$ of $U_q(s\ell_2)$ such that $\mathcal{C}_{2h-p}$ and $\mathcal{V}_{2h-p}$ appear as $U_q(s\ell_2)$ invariant submodules while their duals, $\mathcal{C}_p$ and $\mathcal{V}_p$, are isomorphic to the quotient spaces $\mathcal{D}_p/\mathcal{C}_{2h-p}$ and $\mathcal{W}_p/\mathcal{V}_{2h-p}$, respectively. $\mathcal{W}_p$ coincide with the indecomposable $2h$-dimensional $U_q(s\ell_2)$ modules considered earlier [15–17].

In section 5 after some general remarks about $U_q(s\ell_n)$ we study the case of $n = 3$. We only consider the analogs of the pairs $(\mathcal{V}_p, \mathcal{V}_{2h-p})$ in this case constructing $d_3(h)$-dimensional indecomposable $U_q(s\ell_3)$ modules $\mathcal{V}_{\boldsymbol{p}}$ ($\boldsymbol{p} = (p_{12}, p_{23}),\ p_{i\,i+1} \in \mathbb{N}$) where

$$d_3(h) = \frac{h(h+1)(2h+1)}{6}. \tag{1.2}$$

The dual modules, $\mathcal{V}_{w_{\rm L}\boldsymbol{p}}$, are defined in terms of the longest element, $w_{\rm L} = w_1 w_2 w_1$, of the $s\ell_3$ Weyl group. We give an explicit construction of the BRS operator

$$Q_{\boldsymbol{p}} :\ \mathcal{V}_{w_{\rm L}\boldsymbol{p}} \to \mathcal{V}_{\boldsymbol{p}} \tag{1.3}$$

using the Poincaré–Birkhoff–Witt (PBW) basis in both modules. For

$$(1 <)\ p_{13} = p_{12} + p_{23} < h \tag{1.4}$$

$Q_{\boldsymbol{p}}$ maps the cosingular vector $|w_{\rm L}\boldsymbol{p}; 0, 0, 0\rangle \in \mathcal{V}_{w_{\rm L}\boldsymbol{p}}$ onto a singular vector in $\mathcal{V}_{\boldsymbol{p}}$ which belongs to the canonical (Lusztig–Kashiwara) basis. Denoting by $\boldsymbol{h}$ the weight $(p_{12} = h,\ p_{23} = h)$; the BRS property is expressed by the relation

$$Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}}\, Q_{\boldsymbol{p}} = 0 \qquad (Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}} : \mathcal{V}_{\boldsymbol{p}} \to \mathcal{V}_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}}). \tag{1.5}$$

The BRS cohomology is found to be trivial (for both $n = 2$ and 3):

$$\text{Ker}\ Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}} = \text{Im}\ Q_{\boldsymbol{p}}\ (\subset \mathcal{V}_{\boldsymbol{p}}) \tag{1.6}$$

Im $Q_{\boldsymbol{p}}$ and Ker $Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}}$ defining an invariant subspace of $\mathcal{V}_{\boldsymbol{p}}$. This invariant subspace is shown to lie in the kernel of the invariant Hermitean form on $\mathcal{V}_{\boldsymbol{p}}$.

2. Preliminaries. Dual representations and (co)singular vectors

We shall assume, for definiteness, throughout this paper that $q = \mathrm{e}^{-\mathrm{i}\frac{\pi}{h}}$ with h an integer greater than n. Such q appears naturally as quantum-group deformation parameter corresponding to the left chiral WZNW field in the conventions of [11]. We shall use the notations $\overline{q} := q^{-1}$, $[m] := \frac{q^m - \overline{q}^m}{q - \overline{q}}$ and $(m)_\pm := \frac{q^{\pm 2m}-1}{q^{\pm 2}-1} \equiv q^{\pm(m-1)}[m]$.

We start by fixing our conventions: $U_q(s\ell_n)$ is a Hopf algebra with $4(n-1)$ (Chevalley) generators $E_i,\ F_i,\ q^{H_i},\ q^{-H_i} \equiv \overline{q}^{H_i}$, $i = 1, 2, \ldots, n-1$ satisfying

$$q^{H_i} E_j = E_j q^{H_i + c_{ij}} \qquad q^{H_i} F_j = F_j q^{H_i - c_{ij}} \qquad q^{H_i} \overline{q}^{H_i} = \mathbb{I} = \overline{q}^{H_i} q^{H_i} \tag{2.1}$$

c_{ij} being the $s\ell_n$ Cartan matrix,

$$[E_i,\ F_j] = \delta_{ij}[H_i] \tag{2.2}$$

and the Serre relations

$$\begin{aligned} &E_{i+1\,i} E_i = q E_i E_{i+1\,i} \qquad \text{for} \quad E_{i+1\,i} := E_i E_{i+1} - q E_{i+1} E_i \quad 1 \leqslant i \leqslant n-2 \\ &F_i F_{i\,i+1} = q F_{i\,i+1} F_i \qquad \text{for} \quad F_{i\,i+1} := F_{i+1} F_i - q F_i F_{i+1} \quad 1 \leqslant i \leqslant n-2. \end{aligned} \tag{2.3}$$

The *coproduct* $\Delta : U_q(s\ell_n) \to U_q(s\ell_n) \otimes U_q(s\ell_n)$ and the *counit* $\varepsilon : U_q(s\ell_n) \to \mathbb{C}$ are algebra homomorphisms fixed by their action on the generators

$$\begin{aligned} &\Delta(q^{\pm H_i}) = q^{\pm H_i} \otimes q^{\pm H_i} \\ &\Delta(E_i) = E_i \otimes q^{H_i} + \mathbb{I} \otimes E_i \qquad \Delta(F_i) = F_i \otimes \mathbb{I} + \overline{q}^{H_i} \otimes F_i \end{aligned} \tag{2.4}$$

$$\varepsilon(q^{\pm H_i}) = 1 \qquad \varepsilon(E_i) = 0 = \varepsilon(F_i). \tag{2.5}$$

The *antipode* $\gamma : U_q(s\ell_n) \to U_q(s\ell_n)$ is an algebra antihomomorphism (i.e. $\gamma(XY) = \gamma(Y)\gamma(X)$) characterized by the property

$$\sum \gamma(X_1) X_2 = \sum X_1 \gamma(X_2) = \varepsilon(X)\, \mathbb{I} \qquad \text{for} \quad \Delta(X) = \sum X_1 \otimes X_2. \tag{2.6}$$

With the above coproduct and counit this relation implies

$$\gamma(q^{\pm H_i}) = q^{\mp H_i} \qquad \gamma(E_i) = -E_i\, \overline{q}^{H_i} \qquad \gamma(F_i) = -q^{H_i} F_i. \tag{2.7}$$

A convenient way to single out the two Borel (Hopf-) subalgebras U_q^F and U_q^E of $U_q(s\ell_n)$ [19] (corresponding to the Gauss decomposition of the WZNW monodromy matrix $M = M_+ M_-^{-1}$ [10]) is to arrange their generators in an $n \times n$ matrix form where

$$\begin{aligned} &M_\pm^{\pm 1} = N_\pm D \qquad D^i_j = q^{d_i} \delta^i_j \\ &\sum_{j=1}^{n} d_j = 0 \qquad \sum_{j=1}^{i} d_j = -\sum_{j=1}^{n-1} (c^{-1})_{ij} H_j \quad i \leqslant n-1 \\ &N_+ = \begin{pmatrix} 1 & (\overline{q}-q)F_1 & (\overline{q}-q)F_{12} & \cdots \\ 0 & 1 & (\overline{q}-q)F_2 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \\ &N_- = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ (\overline{q}-q)E_1 & 1 & 0 & \cdots \\ (\overline{q}-q)E_{21} & (\overline{q}-q)E_2 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \end{aligned} \tag{2.8}$$

(In fact, the weight operators q^{d_i}, together with E_i and F_i, give rise to the so-called *simply connected [20] quantum universal enveloping algebra $U_q(s\ell_n)$*.) The subalgebra U_q^F (resp., U_q^E) is generated by the matrix elements of M_+ (resp., M_-^{-1}). The coproduct of the latter is expressed as the matrix multiplication of two copies of M_+ (resp., M_-) while the antipode is related to the inverse matrices:

$$\Delta(M_\pm{}^\alpha{}_\beta) = M_\pm{}^\alpha{}_\sigma \otimes M_\pm{}^\sigma{}_\beta \qquad (M_-^{-1})^\alpha{}_\beta = \gamma(M_-{}^\alpha{}_\beta). \tag{2.9}$$

We introduce a *grading* in $U_q(s\ell_n)$ (corresponding to the *depth* of [13]) ascribing degree 1 to E_i, 0 to $q^{\pm H_i}$, and -1 to F_i (so that, e.g. E_{21} and F_{12} have degree 2 and -2, respectively); we denote by U_q^- (U_q^+) the subalgebra of U_q^F (U_q^E) of elements of negative (resp., positive) degree.

Next we introduce the concepts of (co)singular vectors and dual indecomposable representations adapting to our case the discussion in section 4 of [13].

Let $\mathcal{V}$ be a $U_q(s\ell_n)$ module. An eigenvector $v \in \mathcal{V}$ of the Cartan generators q^{H_i} (i.e. a *weight vector*) is said to be *(F-) singular* if it satisfies $U_q^- \, v = 0$ (sometimes we shall also use for such a vector the common term 'LW vector'). It is called *cosingular* if there is no $v' \in \mathcal{V}$ such that $v \in U_q^+ v'$. *Two cosingular vectors are equivalent* if their difference belongs to $U_q^+ v_1$ for some $v_1 \in \mathcal{V}$. The LW vector in a Verma module is both singular and cosingular.

These definitions are adapted for LW modules. For HW modules one can just reverse the roles of U_q^+ and U_q^- (and of LW and HW vectors).

We shall be dealing in what follows with *indecomposable* $U_q(s\ell_n)$ *modules* $\mathcal{V}_p$ corresponding—in a way that will be made clear below—to integral dominant weights $\Lambda = p - \rho$, and their duals. Here p are the *shifted* weights and ρ is the half sum of the positive roots; we shall use the (linearly dependent) 'barycentric' coordinates of p

$$p = \left\{ (p_1, \ldots, p_n);\ \sum_{i=1}^n p_i = 0,\ p_{i\,i+1} := p_i - p_{i+1} \in \mathbb{N} = \{1, 2, \ldots\} \right\} \tag{2.10}$$

fixed by the condition $\Lambda \equiv \sum_{i=1}^{n-1} \lambda_i \, \Lambda^{(i)} = \sum_{i=1}^{n-1} (p_{i\,i+1} - 1)\, \Lambda^{(i)}$, where $\Lambda^{(i)}$ are the fundamental weights ($\rho = \sum_{i=1}^{n-1} \Lambda^{(i)}$).

There are several inequivalent representations of this type which share the following properties.

(i) $\mathcal{V}_p$ is a direct sum of finite-dimensional weight spaces

$$\mathcal{V}_p = \oplus_{\lambda \in \Lambda_- + Q} \mathcal{V}_p^{(\lambda)} \qquad (q^{H_i} - q^{\lambda_i})\mathcal{V}_p^{(\lambda)} = 0 \quad (\dim \mathcal{V}_p^{(\lambda)} < \infty) \tag{2.11}$$

where $\Lambda_- = \sum_{i=1}^{n-1} (1 - p_{n-i\, n+1-i})\, \Lambda^{(i)}$ and Q is the $s\ell_n$ root lattice; the LW subspaces are one-dimensional

$$\dim \mathcal{V}_p^{(\Lambda_-)} = 1. \tag{2.12}$$

We shall keep (2.12) as a defining property of $\mathcal{V}_p$ even when $\mathcal{V}_p^{(\Lambda_-)}$ is not a LW subspace, i.e. for $F_i\, \mathcal{V}_p^{(\Lambda_-)} \not\equiv 0$, but the corresponding vector that generates $\mathcal{V}_p^{(\Lambda_-)}$ is cosingular.

(ii) The $U_q(s\ell_n)$ Casimir operators are multiples of the identity in $\mathcal{V}_p$; their eigenvalues are expressed as polynomials in $\overline{q}^{p_i}$ (coinciding with the Casimir eigenvalues for finite-dimensional irreducible subrepresentations). In particular, the second-order Casimir operator and its eigenvalue are related to the $U_q(s\ell_n)$ invariant q-trace of the corresponding monodromy matrix (2.8), $\mathrm{tr}(\overline{q}^{2\rho^\vee} M)$ (cf [19]), where $\overline{q}^{2\rho^\vee} := \overline{q}^{\sum_{\alpha>0} H_\alpha}$ is taken in the fundamental representation, $\overline{q}^{2\rho^\vee} = \mathrm{diag}\,\{q^{1-n}, q^{3-n}, \ldots, q^{n-1}\}$

$$\left\{ \mathrm{tr}\,(\overline{q}^{2\rho^\vee} M) - \sum_{i=1}^n \overline{q}^{2p_i} \right\} \mathcal{V}_p = 0. \tag{2.13}$$

For $n = 2$ we obtain ($p \equiv p_1 - p_2$)

$$\{(\overline{q} - q)^2 FE + \overline{q}^{H+1} + q^{H+1} - q^p - \overline{q}^p\}\, \mathcal{V}_p = 0 \tag{2.14}$$

an equality that can be also cast into the more familiar form

$$\left(C_2 - 2\left[\frac{p-1}{2}\right]\left[\frac{p+1}{2}\right]\right) \mathcal{V}_p = 0 \qquad C_2 = EF + FE + [2]\left[\frac{H}{2}\right]^2. \tag{2.15}$$

For $n = 3$ (2.13) yields the following analogue of (2.14):

$$\begin{aligned}\Bigg\{(\overline{q} - q)^2 \bigg(& F_1\, E_1 \overline{q}^{\frac{H_1+2H_2}{3}+1} + F_{12}\, E_{21} q^{\frac{H_2-H_1}{3}-1} + F_2\, E_2 q^{\frac{2H_1+H_2}{3}+1}\bigg) \\ & + \overline{q}^{\frac{2}{3}(2H_1+H_2)+2} + q^{\frac{2}{3}(H_1-H_2)} + q^{\frac{2}{3}(H_1+2H_2)+2} \\ & - \overline{q}^{2p_1} - \overline{q}^{2p_2} - \overline{q}^{2p_3}\Bigg\}\, \mathcal{V}_p = 0.\end{aligned} \tag{2.16}$$

The *dual space* $\mathcal{V}'_p$—i.e. the space of linear forms $\langle f, \cdot\rangle$ on $\mathcal{V}_p$, carries the contragradient representation $X \to \check{X}$. We shall define it by

$$\langle \check{X} f,\, v\rangle = \langle f,\, \gamma \circ \sigma\,(X) v\rangle \qquad (f \in \mathcal{V}'_p,\ v \in \mathcal{V}_p\,) \tag{2.17}$$

where γ is the antipode (2.7) and σ a U_q automorphism (introduced explicitly for reasons that will be made clear below).

We shall make use of the following result (cf section 3 of [10]).

Proposition 2.1. *The associative algebra $U_q(s\ell_n)$ admits, for q on the unit circle, a linear antiinvolution $X \to X'$ and an antilinear Hermitean conjugation $X \to X^*$ (both preserving the commutation relations) which make the Cartan generators q^{H_i} symmetric, resp. unitary*

$$\left(q^{\pm H_i}\right)' = q^{\pm H_i} \qquad \left(q^{\pm H_i}\right)^* = q^{\mp H_i} \tag{2.18}$$

and extend to a coalgebra homomorphism, resp. antihomomorphism

$$\left(X \otimes Y\right)' = X' \otimes Y' \qquad (X \otimes Y)^* = Y^* \otimes X^*. \tag{2.19}$$

The Hermitean conjugation is uniquely determined from these properties:

$$E_i^* = F_i \qquad F_i^* = E_i. \tag{2.20}$$

The 'transposition' $X \to X'$ is determined by (2.19) and (2.18) up to a cyclic inner automorphism, $E_i \to q^m E_i$, $F_i \to \overline{q}^m F_i$, $m \in \mathbb{Z}$. We shall fix this freedom setting, for the Chevalley generators

$$E'_i = F_i\, q^{H_i - 1} \qquad F'_i = \overline{q}^{H_i - 1} E_i \tag{2.21}$$

a choice yielding a symmetric monodromy matrix.

We shall choose in what follows the automorphism σ so that

$$\gamma \circ \sigma(X) = X' \qquad \forall X \in U_q. \tag{2.22}$$

Using (2.7) and defining the transposition as in (2.21), one can see that for $U_q(s\ell_n)$ this amounts to setting σ equal to the involutive automorphism

$$\sigma(q^{H_i}) = \overline{q}^{H_i} \quad \sigma(E_i) = -q F_i \qquad \sigma(F_i) = -\overline{q} E_i \Rightarrow \sigma^2 = \mathrm{id}. \tag{2.23}$$

The most important feature of the choice (2.22) is that it makes the map $\gamma \circ \sigma$ involutive too, $(\,\gamma \circ \sigma\,)^2 = \mathrm{id}$; the invariance of the pairing on $\mathcal{V}'_p \times \mathcal{V}_p$ can be now expressed as

$$\langle\, \check{X} f,\, v\,\rangle = \langle\, f,\, X' v\,\rangle \qquad \text{for all} \quad f \in \mathcal{V}'_p \quad v \in \mathcal{V}_p \quad X \in U_q. \tag{2.24}$$

As will become explicit in the following sections (see also [14]), in fact the contragradient representation of $\mathcal{V}_p$ is *equivalent* to that of weight

$$p' = w_{\mathrm{L}} p = \{p_n, \ldots, p_1\} \Rightarrow p'_{i\,i+1} = -p_{n-i\,n-i+1} \tag{2.25}$$

with w_{L} being the longest Weyl-group element

$$w_{\mathrm{L}} = w_1 \ldots w_{n-1}\, w_1 \ldots w_{n-2}\, \ldots\, w_1 w_2\, w_1 \tag{2.26}$$

i.e. $\mathcal{V}'_p \simeq \mathcal{V}_{p'}$. We note that the involutive element w_{L} ($w_{\mathrm{L}}^2 = 1$) only coincides with the reflection w_θ with respect to the highest root θ for $n = 2, 3$. Identifying w_{L} with its $(n-1)$-dimensional representation in the basis of fundamental weights $\Lambda^{(i)}$ we find that $\det w_{\mathrm{L}} = (-1)^{\binom{n}{2}}$.

The counterpart of the weight space expansion (2.11) for $\mathcal{V}_{p'}$ reads

$$\mathcal{V}_{p'} = \oplus_{\lambda\in\Lambda_-{}'+Q} \mathcal{V}_{p'}^{(\lambda)} \qquad \dim\, \mathcal{V}_{p'}^{(\lambda)} = \dim\, \mathcal{V}_{p}^{(\lambda)} \tag{2.27}$$

with the (finite-dimensional) space $\mathcal{V}_{p'}^{(\lambda)}$ being equivalent to the dual to $\mathcal{V}_{p}^{(\lambda)}$. It follows from (2.17) that for homogeneous elements $v_\lambda \in \mathcal{V}_{p}^{(\lambda)}$ and $f_\mu \in \mathcal{V}_{p'}^{(\mu)}$ the pairing $\langle f, v\rangle$ is only non-zero if $\mu = \lambda$:

$$\langle f_\mu,\, v_\lambda\rangle = \delta_{\mu\lambda}\, \langle f_\lambda,\, v_\lambda\rangle. \tag{2.28}$$

Remark 2.2. Using the notation $\mathcal{V}_{p'}$ we thus extend the admissible values of p to Weyl-group images of dominant weights which are no longer dominant. We shall adopt the resulting more general label for the class of indecomposable representations of interest in what follows.

For $p_{1n} < h$ there is a unique finite-dimensional *irreducible* $U_q(s\ell_n)$ module $\mathcal{V}_p$ and its dual is the (irreducible) contragradient module $\mathcal{V}_{p'}$. If we allow for indecomposable modules, there is of course a wider list of possibilities even when one restricts attention, as we shall, to finite-dimensional representations. In this more general case dual pairs $(\mathcal{V}_p,\, \mathcal{V}_{p'})$ are characterized by the following relationship among their singular and cosingular vectors.

Proposition 2.3. *There is a duality between singular vectors and equivalence classes of cosingular vectors:*

$$\mathrm{Ker}\,\left(U_q^{\mp}|_{\mathcal{V}_p}\right)' = \mathcal{V}_{p'}/U_q^{\pm}\mathcal{V}_{p'}. \tag{2.29}$$

In other words, the bilinear pairing $\langle , \rangle : \mathcal{V}_{p'} \times \mathcal{V}_p \to \mathbb{C}$ *projects to nondegenerate pairings*

$$\langle , \rangle : \mathcal{V}_{p'}/U_q^{\pm}\mathcal{V}_{p'} \times \mathrm{Ker}\, U_q^{\mp}|_{\mathcal{V}_p} \to \mathbb{C}. \tag{2.30}$$

Proof. The argument is the same as in section 4 of [13]. To fix the ideas, we shall demonstrate that $\mathrm{Ker}\, U_q^-|_{\mathcal{V}_p}$ is the subspace of $\mathcal{V}_p$ orthogonal to $U_q^+\mathcal{V}_{p'}$. Let $v \in \left(U_q^+\mathcal{V}_{p'}\right)^\perp (\subset \mathcal{V}_p)$; then, for all $f \in \mathcal{V}_{p'}$, $X^+ \in U_q^+$

$$0 = \langle \check{X}^+ f,\, v\rangle = \langle f,\, \gamma\circ\sigma(X^+)v\rangle \Rightarrow U_q^- v = 0. \tag{2.31}$$

(In the last implication we have used the non-degeneracy of the form $\langle , \rangle$ and the fact that the map $\gamma\circ\sigma : U_q^+ \to U_q^-$ is onto.) This proves that each vector $v \in (U_q^+\mathcal{V}_{p'})^\perp \subset \mathcal{V}_p$ is singular. The converse statement follows from the fact that the map $\gamma\circ\sigma$ is invertible. □

3. Indecomposable h-dimensional representations of $U_q(s\ell_2)$

We shall study in this section the simplest, rank $r = 1$, case of $U_q(s\ell_2)$ generated by $E, F, q^H, \overline{q}^H$ satisfying $q^H\overline{q}^H = 1 = \overline{q}^H q^H$ and

$$[E, F] = [H] = \frac{q^H - \overline{q}^H}{q - \overline{q}} \qquad q^H E = E q^{H+2} \quad q^H F = F q^{H-2}. \tag{3.1}$$

A major simplification in this case results from the fact that both products EF and FE are expressed in terms of the Cartan generators $q^{\pm H}$ and the central operator $\left[\frac{p}{2}\right]^2$ defined in terms of the quadratic Casimir invariant (2.15), or equivalently

$$EF + \left[\frac{H-1}{2}\right]^2 = \left[\frac{p}{2}\right]^2 = FE + \left[\frac{H+1}{2}\right]^2. \tag{3.2}$$

There are three inequivalent dual pairs of h-dimensional representations of $U_q(s\ell_2)$ corresponding to different 'rational factorizations' of EF and FE. All six representations share the existence of a canonical weight basis v_{pm} such that

$$(q^H - q^{2m-p+1})v_{pm} = 0 \qquad p,\ m \in \mathbb{Z} \tag{3.3}$$

$$(EF - [m][p-m])v_{pm} = 0 = (FE - [m+1][p-m-1])v_{pm}. \tag{3.4}$$

Clearly, the eigenvalues of q^H, EF and FE are periodic in m (with period h) and in p (with period $2h$). We shall require that the action of E and F in the six representations we are going to define preserves this periodicity property.

First, we introduce the lowest (resp. highest) weight *Verma modules* [20,21] $\mathcal{V}_p^-$ ($\mathcal{V}_p^+$) for which the action of E and F on the respective canonical bases $v_{pm} = |p, m\rangle_-$ ($|p, m\rangle_+$) are given by

$$E|p, m\rangle_- = |p, m+1\rangle_- \ \Rightarrow\ F|p, m\rangle_- = [m][p-m]|p, m-1\rangle_- \tag{3.5}$$

$$F|p, m\rangle_+ = |p, m-1\rangle_+ \Rightarrow E|p, m\rangle_+ = [m+1][p-m-1]|p, m+1\rangle_+ \tag{3.6}$$

assuming that the LW vector in $\mathcal{V}_p^-$ is $|p, 0\rangle_-$, and the HW vector in $\mathcal{V}_p^+$ is $|p, h-1\rangle_+$, thus fixing the range of m as $m \geqslant 0$ and $m \leqslant h-1$, respectively.

Using the periodicity property in m, we can introduce the *cyclic* h-dimensional counterparts $\mathcal{C}_p^\pm$ of $\mathcal{V}_p^\pm$ identifying all $|p, m\rangle_\pm$ with $m \bmod h$. It is easy to prove that the representation $\mathcal{C}_p^+$ is equivalent to the dual of $\mathcal{C}_p^-$: in the basis f_{pm} of $(\mathcal{C}_p^-)'$ defined by $\langle f_{pm}, v_{pn}\rangle = \delta_{mn}$, $0 \leqslant m, n \leqslant h-1$, one obtains from (2.24)

$$\check{E} f_{pm} = q^{2-p+2m}[m+1][p-m-1] f_{p\,m+1} \qquad \check{F} f_{pm} = q^{p-2m} f_{p\,m-1}$$

and it only remains to renormalize the basis vectors $f_{pm} \to q^{m(m+1-p)} f_{pm}$ to obtain a full coincidence with (3.6). Note that the coefficients for both $\mathcal{C}_p^-$ and $\mathcal{C}_p^+$ are invariant with respect to the transformation $(p, m) \to (-p, m-p)$.

For the second pair $\mathcal{C}_{\pm p}$ of cyclic representations the canonical-basis vectors $(v_{pm} =)|p, m\rangle \in \mathcal{C}_p$ and $|-p, m-p\rangle \in \mathcal{C}_{-p}$ both satisfy (3.3), (3.4) and

$$E|p, m\rangle = (m+1)_+|p, m+1\rangle \qquad F|p, m\rangle = q^{2-p}(p-m)_+|p, m-1\rangle \tag{3.7}$$

and the counterpart of these relations for negative p (reflecting the symmetry $p \leftrightarrow -p$ in (2.15) and (3.2))

$$\begin{aligned} E\,|-p, m-p\rangle &= (m+1-p)_+|-p, m+1-p\rangle \\ F\,|-p, m-p\rangle &= -q^p\,(m)_-|-p, m-1-p\rangle. \end{aligned} \tag{3.8}$$

Note that one can obtain the $U_q(s\ell_2)$ action in $\mathcal{C}_p$ from that in $\mathcal{C}_p^-$ by a simple renormalization of the canonical-basis vectors defining $|p, m\rangle := \frac{1}{(m)_+!}|p, m\rangle_-$ for $0 \leqslant m \leqslant h-1$. The corresponding extension of this transformation to the Verma module would be singular.

The fact that $\mathcal{C}_{-p}$ is equivalent to the dual of $\mathcal{C}_p$ follows again from an explicit calculation. Indeed, for $v_{pm} = |p, m\rangle \in \mathcal{C}_p$ and $f_{pm} \in (\mathcal{C}_p)'$ such that $\langle f_{pm}, v_{pn}\rangle = \delta_{mn}$, applying (2.24) one finds

$$\check{E} f_{pm} = -q^2(m+1-p)_+ f_{p\,m+1} \qquad \check{F} f_{pm} = q^{p-2}(m)_- f_{p\,m-1}.$$

To see the equivalence of this law with (3.8), it is enough to renormalize $f_{pm} = (-q^2)^m|-p, m-p\rangle$.

We shall be chiefly interested in the third pair of modules whose existence exploits the fact that for vanishing EF and FE there is an additional freedom in choosing E or F. We define from the outset the module $\mathcal{V}_p$ as an h-dimensional vector space spanned by $\{|p, m\rangle,\ 0 \leqslant m \leqslant h-1\}$ such that $|p, 0\rangle$ is a LW vector and $|p, h-1\rangle$ is a HW vector. The basis vectors are assumed to satisfy (3.7) except that $F|p, 0\rangle = 0$. A natural basis for displaying its properties is given by

$$e_{pm} = ([m]!)^{-\frac{1}{2}} E^m |p, 0\rangle \quad \left(= ([m]!)^{\frac{1}{2}} q^{\binom{m}{2}} |p, m\rangle\right)$$

yielding

$$E\, e_{pm} = [m+1]^{\frac{1}{2}} e_{p\,m+1} \qquad F e_{pm} = [m]^{\frac{1}{2}} [p-m] e_{p\,m-1}. \tag{3.9}$$

The change of basis $|p, m\rangle \to e_{pm}$ is non-singular in $\mathcal{V}_p$ since $[m]!$ is positive for $0 \leqslant m \leqslant h-1$. However, the action of F implied by (3.9) would be only recovered from (3.7) if we replace the periodicity condition $|p, m+h\rangle = |p, m\rangle$ by the boundary condition $|p, -1\rangle = 0$. Alternatively, we can set

$$F|p, m\rangle = (1-\delta(m)) q^{2-p}(p-m)_+ |p, m-1\rangle \qquad E|p, m\rangle = (m+1)_+ |p, m+1\rangle \tag{3.10}$$

then $\mathcal{V}_p$ can be continued to a periodic module, too, by extending the Kronecker δ periodically (setting $\delta(m) = 1$ for $m = 0 \bmod h$, $\delta(m) = 0$ otherwise). Such an extension would result, however, in just a direct sum of infinitely many equivalent h-dimensional $U_q(s\ell_2)$ submodules. It is the realization (3.10) which justifies preserving the term 'rational factorization' for the module $\mathcal{V}_p$. It will prove more convenient to use (3.10) in the description of dual modules and 'BRS operators' below. The existence of a non-singular basis satisfying (3.9) makes manifest the consistency of using the artificially looking factor $(1-\delta(m))$ in the action of F (3.9). The module $\mathcal{V}_p$ for $0 < p < h$ is dual to $\mathcal{V}_{2h-p} (\simeq \mathcal{V}_{-p})$ where $\mathcal{V}_{2h-p}$ can be defined as follows:

$$\begin{aligned}
&\mathcal{V}_{2h-p} = \{|2h-p, h+m-p\rangle,\ 0 \leqslant m \leqslant h-1\} \\
&E|2h-p, h+m-p\rangle = (1-\delta(h-m-1))(m+1-p)_+ |2h-p, h+m-p+1\rangle \\
&F|2h-p, h+m-p\rangle = -q^p (m)_- |2h-p, h+m-p-1\rangle.
\end{aligned} \tag{3.11}$$

(The range of values for m corresponds to the target space of the BRS operator Q^{h-p} defined below, see (3.18).) Although equations (3.10) and (3.11) make sense for $0 < p < 2h$, we shall assume $0 < p < h$ for both $\mathcal{V}_p$ and $\mathcal{V}_{2h-p}$, thus avoiding ambiguity. Note that for $p = h$ the two dual modules become equivalent and irreducible.

Remark 3.1. The modules $\mathcal{C}_p$ ($p \in \mathbb{Z}$) are, unlike $\mathcal{C}_p^{\pm}$ and $(\mathcal{V}_p,\ \mathcal{V}_{2h-p})$, self-conjugate in the following sense. There exists an antiunitary operator Θ in $\mathcal{C}_p$ of unit square which implements the Hermitean conjugation (2.20). It acts on the canonical basis according to

$$\Theta\, |p, m\rangle = q^{m(m+1-p)}\, |p, p-1-m\rangle. \tag{3.12}$$

In verifying its properties we use the antilinearity of Θ; for example

$$\begin{aligned}
\Theta E \Theta |p, m\rangle &= \Theta q^{m(m+1-p)} (p-m)_+ |p, p-m\rangle \\
&= \overline{q}^{m(m+1-p)} \overline{(p-m)_+} q^{(p-m)(1-m)} |p, m-1\rangle
\end{aligned}$$

$$\begin{aligned} &= q^{2-p}(p-m)_+|p, m-1\rangle = F|p,m\rangle \\ \Theta q^H \Theta|p,m\rangle &= q^{m(p-1-m)}\Theta q^{p-1-2m}|p, p-1-m\rangle \\ &= q^{2m-p+1}|p,m\rangle = q^H|p,m\rangle. \end{aligned} \tag{3.13}$$

In particular, Θ exchanges the HW vector $|p, h-1\rangle$ with the LW one, $|p, p\rangle$ ($\equiv |p, p-h\rangle$). There is no operator with these properties mapping $\mathcal{C}_p^+$ ($\mathcal{C}_p^-$) into itself since $\mathcal{C}_p^+$ ($\mathcal{C}_p^-$) only admits highest (lowest) weight vectors. One can, however, define a Θ which intertwines $\mathcal{C}_p^+$ with $\mathcal{C}_p^-$.

The representations described above are characterized by the structure of their singular and cosingular vectors. The cyclic representation $\mathcal{C}_p^-$ contains two singular vectors, $|p, 0\rangle_-$ and $|p, p\rangle_-$ and no cosingular ones, whereas in $\mathcal{C}_p^+$ there are two cosingular, $|p, 0\rangle_+$ and $|p, p\rangle_+$, and no singular vectors. Both representations are therefore irreducible.

The h-dimensional cyclic representations $\mathcal{C}_{\pm p}$ and the modules $\mathcal{V}_p$, $\mathcal{V}_{2h-p}$ are, on the other hand, indecomposable. Namely, $\mathcal{C}_p$ and $\mathcal{V}_p$ admit an $(h-p)$-dimensional invariant subspace $\mathcal{I}_{h-p}$ while $\mathcal{C}_{2h-p}$ and $\mathcal{V}_{2h-p}$ contain a p-dimensional invariant subspace $\mathcal{I}_p$. The lowest and HW vectors of the (irreducible) invariant subspaces of $\mathcal{C}_p$ and $\mathcal{C}_{2h-p}$ are

$$\{|p, p\rangle, \ |p, h-1\rangle\} \in \mathcal{I}_{h-p} \qquad \{|2h-p, h-p\rangle, \ |2h-p, h-1\rangle\} \in \mathcal{I}_p \tag{3.14}$$

respectively; the vectors $|\pm p, \pm p\rangle$ are the only singular vectors in $\mathcal{C}_{\pm p}$, and $|\pm p, 0\rangle$ are the cosingular ones. The labels of $\mathcal{C}_{\pm p}$ can be given now the following meaning: they are equal to the eigenvalue (mod $2h$) of the operator $1-H$ on the cosingular vector or, equivalently, of $H-1$ applied to the corresponding singular vector (cf equations (3.3) and (3.14)).

It follows from (3.7) and (3.8) that the operators E and F are nilpotent in both $\mathcal{C}_{\pm p}$, and $\mathcal{V}_p$, $\mathcal{V}_{2h-p}$; for example

$$E^h = 0 = F^h \ (= q^{hH} - (-1)^{p-1}) \tag{3.15}$$

and identical relations are true for $\check{E}$, $\check{F}$ and $q^{\check{H}}$.

The irreducible $U_q(s\ell_2)$ representation in $\mathcal{I}_{h-p}$ is equivalent to the one in the quotient $\mathcal{C}_{2h-p}/\mathcal{I}_p$ or $\mathcal{V}_{2h-p}/\mathcal{I}_p$ and vice versa. The partial equivalence of $\mathcal{C}_p$ ($\mathcal{V}_p$) with $\mathcal{C}_{-p} \simeq \mathcal{C}_{2h-p}$ ($\mathcal{V}_{-p} \simeq \mathcal{V}_{2h-p}$) will be displayed by explicitly constructing the corresponding intertwining maps Q^p and Q^{h-p} which can be characterized by their invariance properties

$$XQ^p = Q^p\check{X} \qquad \check{X}Q^{h-p} = Q^{h-p}X \qquad \forall X \in U_q(s\ell_2). \tag{3.16}$$

Proposition 3.2. *The maps $Q^p : \mathcal{C}_{-p} \to \mathcal{C}_p$ and $Q^{h-p} : \mathcal{C}_p \to \mathcal{C}_{2h-p}$ are determined up to m-independent factors from the $U_q(s\ell_2)$ invariance conditions (3.16) and are given by*

$$Q^p\,|-p, m-p\rangle = \frac{(m)_+!}{(m-p)_+!}\,|p,m\rangle \tag{3.17}$$

$$Q^{h-p}\,|p,m\rangle = \frac{(h+m-p)_+!}{(m)_+!}\,|2h-p, h+m-p\rangle. \tag{3.18}$$

They satisfy

$$\mathrm{Ker}\ Q^p = \mathcal{I}_p = \mathrm{Im}\ Q^{h-p} \quad (\mathcal{C}_{-p} \simeq \mathcal{C}_{2h-p}) \qquad \mathrm{Im}\ Q^p = \mathcal{I}_{h-p} = \mathrm{Ker}\ Q^{h-p}. \tag{3.19}$$

Furthermore, Q^p is the pth and Q^{h-p}—the $(h-p)$th power of an operator Q acting on $\oplus_{p\in\mathbb{Z}}\mathcal{C}_p$ as

$$Q|p,m\rangle = (m+1)_+|p+2, m+1\rangle. \tag{3.20}$$

The BRS operators (3.17) and (3.18) also act on $\mathcal{V}_p$, $\mathcal{V}_{-p}$:

$$Q^p : \mathcal{V}_{-p} \to \mathcal{V}_p \qquad Q^{h-p} : \mathcal{V}_p \to \mathcal{V}_{2h-p}. \tag{3.21}$$

Alternatively, one can define the intertwiners between the dual pairs of representations using the following fact.

Proposition 3.3. *There exist (unique up to normalization) bilinear forms* $(\Phi|\Psi)$ *on* $\mathcal{C}_{\pm p}$ *and on* $\mathcal{V}_{\pm p}$ *for* $0 < p < h$ *invariant with respect to the transposition* $X \to X'$ *of proposition 2.1*

$$(\Phi|X\,\Psi) = (X'\,\Phi|\Psi). \tag{3.22}$$

The form on $\mathcal{C}_p$ *and on* $\mathcal{V}_p$ *vanishes on the* $(h-p)$*-dimensional* $U_q(s\ell_2)$ *invariant subspace* $\mathcal{I}_{h-p}$ *with lowest and HW vectors* $|p, p\rangle$ *and* $|p, h-1\rangle$*, respectively, while*

$$\begin{aligned}&(p, m|p, m') = \frac{N_p\,\delta_{mm'}}{(m)_+!(p-m-1)_-!}\\ &\text{for}\quad 0 \leqslant m \leqslant p-1 \qquad \text{or} \qquad 0 \leqslant m' \leqslant p-1.\end{aligned} \tag{3.23}$$

It satisfies the conjugation property

$$\overline{(p, m|p, m)} = (p, p-m-1|p, p-m-1) \qquad \text{for}\quad N_p = \overline{N_p}. \tag{3.24}$$

The corresponding relations for $\mathcal{C}_{-p}$ *and* $\mathcal{V}_{-p}$ *are*

$$\begin{aligned}&(-p, m-p|-p, m'-p) = \frac{N_{-p}\,\delta_{mm'}}{(h-1-m)_-!(m-p)_+!}\\ &(= 0 \text{ for } 0 \leqslant m \leqslant p-1 \text{ or } 0 \leqslant m' \leqslant p-1,\ \text{i.e. on } \mathcal{I}_p)\end{aligned} \tag{3.25}$$

and, for $p \leqslant m \leqslant h-1$

$$\begin{aligned}&\overline{(-p, h-1-m|-p, h-1-m)} = (-p, m-p|-p, m-p)\\ &\text{for}\quad N_{-p} = \overline{N_{-p}}.\end{aligned} \tag{3.26}$$

(Note that in (3.22), in contrast to (2.24), X' is in the *same* representation as X, not in the contragradient one.)

Proof. The orthogonality for $m \neq m'$ follows from $q^H = (q^H)'$. Applying (3.22) say, for $\Phi = |p, m+1\rangle$, $\Psi = |p, m\rangle$ (resp., $\Phi = |-p, m-p+1\rangle$, $\Psi = |-p, m-p\rangle$) and $X = E$, one obtains recurrence relations solved by (3.23) and (3.25), respectively. □

The (degenerate) invariant bilinear forms on $\mathcal{C}_p$ and $\mathcal{C}_{-p}$ are related, through the intertwiners Q^{h-p} and Q^p, to the pairing between dual modules. One can prove the following relations that can be used alternatively as a definition of the intertwiners:

$$\langle f_1,\ Q^p f_2\rangle = M_{-p}\,(f_1|f_2) \qquad \langle Q^{h-p} v_1,\ v_2\rangle = M_p\,(v_1|v_2) \tag{3.27}$$

where $v_1, v_2 \in \mathcal{C}_p,\ f_1, f_2 \in \mathcal{C}_{-p}$ and $M_{\pm p}$ are m-independent.

We can now prove the equivalence of (3.16) and (3.27):

$$\begin{aligned}\langle f_1,\ X\,Q^p\,f_2\rangle &= \langle \check{X}'\,f_1,\ Q^p\,f_2\rangle = M_{-p}\,(\check{X}'\,f_1\,|\,f_2)\\ &= M_{-p}\,(f_1|\check{X}\,f_2) = \langle f_1,\ Q^p\,\check{X}\,f_2\rangle\end{aligned} \tag{3.28}$$

$$\begin{aligned}\langle \check{X}\,Q^{h-p}\,v_1,\ v_2\rangle &= \langle Q^{h-p}\,v_1,\ X'\,v_2\rangle = M_p\,(v_1|X'\,v_2)\\ &= M_p\,(X\,v_1|v_2) = \langle Q^{h-p}\,X\,v_1,\ v_2\rangle.\end{aligned} \tag{3.29}$$

We have used (2.24), (3.27) and the involutivity of the transposition (2.19) (note also that $(X')\check{} = (\check{X})' \equiv \check{X}'$).

From (3.17), (3.18), (3.27) and (3.32) it follows that

$$\langle -p, m-p|p, m\rangle = (-1)^m q^{1-p+2m} N_{-p} \equiv (-1)^m (\overline{q}^{2\rho^\vee})^m_m N_{-p} \tag{3.30}$$

$$\langle 2h-p, h+m-p|p, m\rangle = (-1)^{m-p} q^{1-p+2m} N_p \equiv (-1)^{m-p} (\overline{q}^{2\rho^\vee})^m_m N_p \tag{3.31}$$

for

$$M_p = q^{p-h-1}\,(h-1)_+! \qquad M_{-p} = q^{1-p}\,(h-1)_-!. \tag{3.32}$$

To derive (3.30)–(3.32) one uses relations (3.17), (3.18) and

$$(h-k)_{\pm}! = \frac{(h-1)_{\pm}!(-q^{\pm 2})^{k-1}}{(k-1)_{\mp}!}. \tag{3.33}$$

Note that in the limit $q \to 1$ one gets the well known factor $(-1)^m$ relating the covariant and the contravariant canonical bases of $s\ell_2$ modules ([22], equations (9)–(125)).

Equating the two expressions one obtains the relation

$$N_{-p} = (-1)^p N_p \tag{3.34}$$

consistent with the reality of $N_{\pm p}$.

Remark 3.4. The modules $\mathcal{C}_p$ and $\mathcal{V}_p$ also admit an invariant Hermitean form obtained by using the Hermitean conjugation $X \to X^*$ of proposition 2.1 instead of the transposition. It can be demonstrated to be positive semidefinite and to majorize the bilinear form. There is a natural choice of normalization for which the (Hermitean) norm square of the basis vectors is given by

$$\||p,m\rangle\|^2 = |(p,m|p,m)| = \frac{|N_p|}{[m]![p-m-1]!} \qquad \text{for} \quad 0 < p \leqslant h. \tag{3.35}$$

The following statement allows one to interpret the operator Q as a generalized exterior derivative [23] or 'BRS operator' [24] with a trivial cohomology.

Proposition 3.5. *The operator Q (3.20) satisfies*

$$Q^h = 0 = Q^p Q^{h-p} = Q^{h-p} Q^p \tag{3.36}$$

so that the sequence of $U_q(s\ell_2)$ modules and intertwiners

$$\cdots \xrightarrow{Q^p} \mathcal{C}_{p-2h} \xrightarrow{Q^{h-p}} \mathcal{C}_{-p} \xrightarrow{Q^p} \mathcal{C}_p \xrightarrow{Q^{h-p}} \mathcal{C}_{2h-p} \xrightarrow{Q^p} \cdots \tag{3.37}$$

is a complex. Furthermore, it is exact, i.e. all cohomology groups are trivial (see equation (3.19)).

In fact, the triviality of the cohomologies is, according to [23], a consequence of the existence of an operator $K : \mathcal{C}_{p+2} \to \mathcal{C}_p$ which, together with Q (3.20) satisfies

$$KQ - q^2 QK = \mathbb{I} \qquad K^h = 0. \tag{3.38}$$

It is given by

$$K|p+2, m+1\rangle = (1 - \delta_{h-1\, m\, (\mathrm{mod}\, h)})|p,m\rangle \quad \big(\Rightarrow K|p,0\rangle = 0\big). \tag{3.39}$$

4. $2h$-dimensional indecomposable representations of $U_q(s\ell_2)$)

The question we address in this section is: can we use the intertwining map $Q^{h-p} : \mathcal{C}_p \to \mathcal{C}_{2h-p}$ to derive raising and lowering operators $\mathfrak{e}$ and $\mathfrak{f}$ on $\mathcal{C}_p \oplus \mathcal{C}_{2h-p}$ of the form

$$\mathfrak{e} = E + \alpha(q^{\check{H}}, q^p)\check{E} Q^{h-p} \oplus \check{E} \qquad \mathfrak{f} = F + \beta(q^{\check{H}}, q^p)\check{F} Q^{h-p} \oplus \check{F} \tag{4.1}$$

$q^{\pm\mathfrak{h}} = q^{\pm(H \oplus \check{H})}$, where, for X standing for an endomorphism of $\mathcal{C}_p$ which represents an element of $U_q(s\ell_2)$, $\check{X}$ stands for the corresponding endomorphism of $\mathcal{C}_{2h-p}$? This would equip the direct sum $\mathcal{C}_p \oplus \mathcal{C}_{2h-p}$ with the structure of an indecomposable $U_q(s\ell_2)$ module.

It turns out that the problem, as stated, has no solution. Its study, however, did lead us to a (slightly more general) construction of a $2h$-dimensional indecomposable $U_q(s\ell_2)$ module $\mathcal{D}_p$ such that $\mathcal{C}_{2h-p}$ appears as an invariant subspace of $\mathcal{D}_p$ while $\mathcal{C}_p$ is isomorphic to the quotient space $\mathcal{D}_p/\mathcal{C}_{2h-p}$. Trying to interpret this result as a realization of the construction (4.1), we would find out that the coefficients α and β are singular functions of their arguments whose poles, however, are compensated by zeros in the kernel of Q^{h-p}. We thus end up with operators $\mathfrak{e}$ and $\mathfrak{f}$ of a form suggested but not literally given by (4.1). Let us reproduce this heuristic argument.

We look for functions $\alpha_m(p)$ and $\beta_m(p)$ such that the operators $\mathfrak{e}$ and $\mathfrak{f}$, acting on the canonical basis in $\mathcal{C}_p$ as

$$\begin{aligned}\mathfrak{e}|p,m\rangle &= (m+1)_+|p,m+1\rangle + \alpha_m(p)\frac{(h+m-p+1)_+!}{(m)_+!}|2h-p,m-p+1\rangle\\ \mathfrak{f}|p,m\rangle &= -q^{p+2-2m}\left((m-p)_+|p,m-1\rangle + \beta_m(p)\frac{(h+m-p)_+!}{(m-1)_+!}|2h-p,m-p-1\rangle\right)\end{aligned} \tag{4.2}$$

satisfy the commutation relation $([\mathfrak{e},\mathfrak{f}]-[\mathfrak{h}])\,\mathcal{C}_p = 0$. This yields recurrence relations for α_m $(=\alpha_m(p))$ and β_m

$$q^2(\alpha_{m-1}+\beta_m)(m)_+(m-p)_+ = (\alpha_m+\beta_{m+1})(m+1)_+(m-p+1)_+. \tag{4.3}$$

These equations have a singular solution:

$$\alpha_m = \frac{\alpha(p)}{[m+1][m-p+1]}\left(= \frac{q^{2m-p}\alpha(p)}{(m+1)_+(m-p+1)_+}\right) \qquad \beta_m = \frac{\beta(p)}{[m][m-p]}. \tag{4.4}$$

The product of α_m and β_m with the ratio of factorials in (4.2) can, however, be given an unambiguous meaning; we obtain

$$\begin{aligned}\mathfrak{e}|p,m\rangle &= (m+1)_+|p,m+1\rangle + \alpha(p)q^{2m-p}\frac{(h+m-p)_+!}{(m+1)_+!}|2h-p,m-p+1\rangle\\ \mathfrak{f}|p,m\rangle &= q^{2-p}(p-m)_+|p,m-1\rangle - \beta(p)\frac{(h+m-p-1)_+!}{(m)_+!}|2h-p,m-p-1\rangle.\end{aligned} \tag{4.5}$$

The ratio in the second terms can be defined for a vanishing denominator using the following general formula for q-binomial coefficients at roots of 1 ([25]):

$$\binom{m_0+hm_1}{n_0+hn_1}_+ = \binom{m_0}{n_0}_+\binom{m_1}{n_1}_{q=1} \qquad \text{for} \quad m_1\in\mathbb{Z} \quad n_1\in\mathbb{Z}_+ \quad 0\leqslant m_0 \quad n_0\leqslant h-1. \tag{4.6}$$

This gives, for example, $\frac{(2h-1-p)_+!}{(h)_+!} = (h-p-1)_+!$ for $0<p<h$. Then the second term in the expression for $\mathfrak{e}|p,m\rangle$ vanishes for $p\leqslant m\leqslant h-2$. For $p=h$ the second term in the right-hand side of both equations (4.5) should disappear. A convenient choice for $\alpha(p)$ and $\beta(p)$ which satisfies this condition is

$$\alpha(p) = \frac{1}{(h-p-1)_+!} = \beta(p). \tag{4.7}$$

Inserting this into (4.5) we find

$$\begin{aligned}\mathfrak{e}\,|p,m\rangle &= (m+1)_+\,|p,m+1\rangle + q^{2m-p}\binom{h+m-p}{m+1}_+|2h-p,m-p+1\rangle\\ \mathfrak{f}\,|p,m\rangle &= q^{2-p}\,(p-m)_+\,|p,m-1\rangle - \binom{h+m-p-1}{m}_+|2h-p,m-p-1\rangle\end{aligned} \tag{4.8}$$

where $\binom{N}{n}_+$ is the q-binomial coefficient $\frac{(N)_+!}{(n)_+!(N-n)_+!}$ which is defined to vanish for $N<n$. Completing (4.8) with

$$\begin{aligned}\mathfrak{e}\,|2h-p,m-p\rangle &= \check{E}|2h-p,m-p\rangle = (m+1-p)_+|2h-p,m+1-p\rangle\\ \mathfrak{f}\,|2h-p,m-p\rangle &= \check{F}|2h-p,m-p\rangle = -q^p(m)_-|2h-p,m-p-1\rangle\end{aligned} \tag{4.9}$$

we find

$$
\begin{aligned}
&(\mathfrak{k} - [m+1][h-m-1]) \begin{pmatrix} |p,m\rangle \\ |2h-p, m-p\rangle \end{pmatrix} \\
&\qquad = \begin{pmatrix} 0 & -2(h-p)_+\binom{h+m-1}{m}_+ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |p,m\rangle \\ |2h-p, m-p\rangle \end{pmatrix} \\
&(\mathfrak{k} - [m+1][h-m-1])^2 \mathcal{D}_p = 0.
\end{aligned}
\tag{4.10}
$$

We thus arrive at the following result.

Proposition 4.1. *The* $2h$*-dimensional vector space* $\mathcal{D}_p$ *with basis*

$$\{|p,m\rangle \in \mathcal{C}_p,\ |2h-p, m-p\rangle \in \mathcal{C}_{2h-p},\ m \bmod h\}$$

equipped with the $U_q(s\ell_2)$ *action (4.8), (4.9) is an indecomposable* $U_q(s\ell_2)$ *module with the following chain of invariant subspaces:*

$$\mathcal{I}_p \subset \mathcal{C}_{2h-p} \subset \tilde{\mathcal{C}}_{2h-p} \subset \mathcal{D}_p. \tag{4.11}$$

Here $\mathcal{I}_p$ *is the* p*-dimensional invariant subspace of the* h*-dimensional cyclic subrepresentation* $\mathcal{C}_{2h-p}$ *with a lowest and a HW vector,* $|2h-p, h-p\rangle$ *and* $|2h-p, h-1\rangle$*, respectively.* $\tilde{\mathcal{C}}_{2h-p}$ *is spanned by* $\mathcal{C}_{2h-p}$ *and* $\mathcal{I}_{h-p}$*. We further have the following identifications (in the notation of section 3) of the cyclic representation* $\mathcal{C}_p$ *and of the irreducible subquotient* $\mathcal{C}_p/\mathcal{I}_{h-p}$*:*

$$\mathcal{I}_{h-p} \subset \mathcal{C}_p \simeq \mathcal{D}_p/\mathcal{C}_{2h-p} \qquad \mathcal{C}_p/\mathcal{I}_{h-p} \simeq \mathcal{D}_p/\tilde{\mathcal{C}}_{2h-p}. \tag{4.12}$$

The Casimir invariant C_2 *(2.15) is indecomposable in* $\mathcal{D}_p$*, the analogue of the first equation (2.15) being*

$$\left(C_2 - 2\left[\frac{p-1}{2}\right]\left[\frac{p+1}{2}\right]\right)^2 \mathcal{D}_p = 0. \tag{4.13}$$

The same construction applies to the pair $(\mathcal{C}_{-p}, \mathcal{C}_p)$ *which is combined in an indecomposable module* $\mathcal{D}_{-p}$ *with composition series* $\mathcal{I}_{h-p} \subset \mathcal{C}_p \subset \tilde{\mathcal{C}}_p \subset \mathcal{D}_{-p}$ *where* $\tilde{\mathcal{C}}_p$ *is* $(h+p)$*-dimensional and* $\mathcal{C}_{-p} \simeq \mathcal{D}_{-p}/\mathcal{C}_p$*.*

Remark 4.2. There are several inequivalent $2h$-dimensional indecomposable $U_q(s\ell_2)$ modules. For instance, one can choose either $\alpha(p)$ or $\beta(p)$ equal to zero. The modules $\mathcal{D}_p$ are singled out as being more symmetric. Indeed, it is only for $\mathcal{D}_p$ that one can extend the antiunitary operator Θ of remark 3.1.

Remark 4.3. One can define in a similar way $2h$-dimensional indecomposable representations $\mathcal{W}_p$ combining the pairs $(\mathcal{V}_p, \mathcal{V}_{2h-p})$. The Fröhlich–Kerler construction ([17], section 5.3) is then reproduced by taking $\beta(p) = 0$, $\alpha(p) \neq 0$.

Concluding the $n = 2$ case, we would like to emphasize the following. As pointed out in the Introduction, the motivation of [15–17] for studying indecomposable $2h$-dimensional $U_q(s\ell_2)$ representations has been their appearance in the tensor-product decomposition of physical ones. (A general study of tensor-product expansions of IR of $U_q(s\ell_2)$ for q a root of unity is contained in [18].) We are advocating here the opposite point of view, introducing indecomposable modules from the outset. The physical ones then appear as appropriate subquotients.

5. Generalization to $U_q(s\ell_n)$. Indecomposable $U_q(s\ell_3)$ modules

One way to write down the canonical basis $\{|p,m\rangle,\ 0 \leqslant m \leqslant h-1\}$ in $\mathcal{V}_p$ that readily generalizes to $U_q(s\ell_n)$ consists of acting by a basis of raising operators of the enveloping algebra on the LW vector $|p,0\rangle$

$$|p,m\rangle = E^{(m)}\,|p,0\rangle \qquad E^{(m)} := \frac{E^m}{(m)_+!} \quad (\text{for } m < h) \quad F|p,0\rangle = 0. \quad (5.1)$$

For $U_q(s\ell_n)$ a straightforward extension of (5.1) is provided by substituting for $\{E^{(m)}\}$ the PBW basis in the subalgebra U_q^+ of raising operators. It is labelled by $\binom{n}{2}$ quantum numbers $m_{ij},\ 1 \leqslant i \leqslant j \leqslant n-1$, the powers of E_i ($\equiv E_{ii}$) and E_{ji}; here E_{ji} is defined by continuing inductively the definition (2.3) of $E_{i+1\,i}$

$$E_{j+1\,i} = E_{ji}\,E_{j+1} - q\,E_{j+1}\,E_{ji} \qquad \text{for} \quad 1 \leqslant i \leqslant j \leqslant n-2. \qquad (5.2)$$

The case $n = 3$ is representative, on one hand, since it shares the main complication in the passage from $n = 2$ to $n > 2$, i.e., the appearance of weights of multiplicity higher than 1; on the other hand, it still allows an explicit description since the 'canonical' [25] (or 'crystal' [26]) basis consists of monomials in E_i just for $n \leqslant 3$. We shall, therefore, proceed in extending the main results of section 3 to this case.

5.1. Finite-dimensional factor algebra of the Borel subalgebra U_q^E. LW $U_q(s\ell_3)$ modules

U_q^E can be viewed as a bigraded associative algebra

$$\begin{array}{llll} U_q^E = \oplus_{\lambda_1,\lambda_2\in\mathbb{Z}_+} U_q^E(\lambda_1,\lambda_2) & U_q^E(\lambda_1,\lambda_2) = \mathrm{Span}\,\{q^{m_1H_1+m_2H_2}E_1^\alpha E_{21}^\beta E_2^\gamma\} & & \\ \alpha,\ \beta,\ \gamma\ \in \mathbb{Z}_+ \quad \alpha+\beta=\lambda_1 & \beta+\gamma=\lambda_2 & m_a\in\mathbb{Z} & \end{array} \qquad (5.3)$$

so that $U_q^E(0,0)$ is the group algebra of the Cartan subgroup of $U_q(s\ell_3)$ generated by $q^{\pm H_a},\ a = 1, 2$. The linear span of the *PBW basis* $\{E_1^\alpha E_{21}^\beta E_2^\gamma\}$ in each $U_q^E(\lambda_1,\lambda_2)$ is taken with operator-valued coefficients belonging to $U_q^E(0,0)$ and satisfying

$$q^{H_a}\,U_q^E(\lambda_1,\lambda_2)\,\overline{q}^{H_a} = q^{3\lambda_a-\lambda_1-\lambda_2}\,U_q^E(\lambda_1,\lambda_2) \qquad a = 1,2. \qquad (5.4)$$

For each dominant weight $\boldsymbol{p} = (p_{12}, p_{23})$ we define an $U_q(s\ell_3)$ module $\mathcal{V}_{\boldsymbol{p}}$ by

$$\mathcal{V}_{\boldsymbol{p}} = U_h^E|\boldsymbol{p};0,0,0\rangle \qquad U_h^E = \oplus_{\lambda_1,\lambda_2=0}^{h-1} U_q^E(\lambda_1,\lambda_2) \qquad F_a|\boldsymbol{p};0,0,0\rangle = 0 \qquad (5.5)$$

where $|\boldsymbol{p};0,0,0\rangle$ is the LW vector in the PBW basis defined by

$$\begin{array}{l} |\boldsymbol{p};\alpha,\beta,\gamma\rangle = E_1^{(\alpha)}E_{21}^{[\beta]}E_2^{(\gamma)}\,|\boldsymbol{p};0,0,0\rangle \\ \alpha,\beta,\gamma \geqslant 0 \qquad \alpha+\beta \leqslant h-1 \qquad \beta+\gamma \leqslant h-1 \end{array} \qquad (5.6)$$

with $X^{[\beta]} \equiv \frac{X^\beta}{[\beta]!}$. The normalization of the basis vectors is chosen in such a way that the periodicity of the eigenvalues of the Cartan generators

$$(q^{H_1} - q^{2\alpha+\beta-\gamma-p_{23}+1})|\boldsymbol{p};\alpha,\beta,\gamma\rangle = 0 = (q^{H_2} - q^{-\alpha+\beta+2\gamma-p_{12}+1})|\boldsymbol{p};\alpha,\beta,\gamma\rangle \qquad (5.7)$$

summed up in the substitution rule

$$(\alpha,\beta,\gamma) \to (\alpha+\varepsilon_1 h, \beta+\varepsilon_2 h, \gamma+\varepsilon_3 h) \qquad \varepsilon_i = \pm 1 \quad i = 1,2,3 \qquad (5.8)$$

corresponds to the periodicity of the expressions for the action of E_a and F_a ($a = 1, 2$) on the PBW basis

$$\begin{aligned} E_1\,|\boldsymbol{p};\alpha,\beta,\gamma\rangle &= (\alpha+1)_+\,|\boldsymbol{p};\alpha+1,\beta,\gamma\rangle \\ E_2\,|\boldsymbol{p};\alpha,\beta,\gamma\rangle &= \overline{q}^{\alpha}\,(q^{\beta}\,(\gamma+1)_+\,|\boldsymbol{p};\alpha,\beta,\gamma+1\rangle \\ &\quad -(1-\delta_{\alpha 0})[\beta+1]\,|\boldsymbol{p};\alpha-1,\beta+1,\gamma\rangle) \\ E_{21}\,|\boldsymbol{p};\alpha,\beta,\gamma\rangle &= q^{\alpha}\,[\beta+1]\,|\boldsymbol{p};\alpha,\beta+1,\gamma\rangle \end{aligned} \qquad (5.9)$$

$$\begin{aligned}
F_1\,|\boldsymbol{p};\alpha,\beta,\gamma\rangle\ &= \overline{q}^{1+\alpha}[p_{23}-\alpha-\beta+\gamma]\,(1-\delta_{\alpha 0})|\boldsymbol{p};\alpha-1,\beta,\gamma\rangle\\
&\quad + q^{\beta-p_{23}}[\gamma+1](1-\delta_{\beta 0})\,|\boldsymbol{p};\alpha,\beta-1,\gamma+1\rangle\\
F_2\,|\boldsymbol{p};\alpha,\beta,\gamma\rangle\ &= q^{1-\gamma}[p_{12}-\gamma](1-\delta_{\gamma 0})\,|\boldsymbol{p};\alpha,\beta,\gamma-1\rangle\\
&\quad - q^{p_{12}-2\gamma}(\alpha+1)_+(1-\delta_{\beta 0})\,|\boldsymbol{p};\alpha+1,\beta-1,\gamma\rangle\\
F_{12}\,|\boldsymbol{p};\alpha,\beta,\gamma\rangle\ &= \overline{q}^{\alpha}[p_{13}-\alpha-\beta-\gamma-1](1-\delta_{\beta 0})\,|\boldsymbol{p};\alpha,\beta-1,\gamma\rangle\\
&\quad - q^{p_{23}+1-2\alpha-\beta}[\gamma-p_{12}]\,(1-\delta_{\alpha 0})(1-\delta_{\gamma 0})|\boldsymbol{p};\alpha-1,\beta,\gamma-1\rangle.
\end{aligned} \tag{5.10}$$

The simplest way to verify the consistency of these relations is to use, following the discussion of the $n = 2$ case (equations (3.9)), the non-periodic (but still regular) basis

$$e_{\boldsymbol{p}}(\alpha,\beta,\gamma) = \big([\alpha+\beta]![\beta+\gamma]!\big)^{-\frac{1}{2}}\,E_1^{\alpha}E_{21}^{\beta}E_2^{\gamma}\,|\boldsymbol{p};0,0,0\rangle \tag{5.11}$$

such that

$$\begin{aligned}
E_1\,e_{\boldsymbol{p}}(\alpha,\beta,\gamma) &= \sqrt{[\alpha+\beta]}\,e_{\boldsymbol{p}}(\alpha+1,\beta,\gamma)\\
E_2 e_{\boldsymbol{p}}(\alpha,\beta,\gamma) &= \sqrt{[\beta+\gamma]}(q^{\beta-\alpha}e_{\boldsymbol{p}}(\alpha,\beta,\gamma+1)-\overline{q}[\alpha]e_{\boldsymbol{p}}(\alpha-1,\beta+1,\gamma))
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
F_1\,e_{\boldsymbol{p}}(\alpha,\beta,\gamma) &= \overline{q}^2\,\frac{[\alpha]}{\sqrt{[\alpha+\beta]}}\,[p_{23}-\alpha-\beta+\gamma]\,e_{\boldsymbol{p}}(\alpha-1,\beta,\gamma)\\
&\quad + q^{\beta-\gamma-p_{23}}\frac{[\beta]}{\sqrt{[\alpha+\beta]}}\,e_{\boldsymbol{p}}(\alpha,\beta-1,\gamma+1)\\
F_2\,e_{\boldsymbol{p}}(\alpha,\beta,\gamma) &= \frac{[\gamma]}{\sqrt{[\beta+\gamma]}}\,[p_{12}-\gamma]\,e_{\boldsymbol{p}}(\alpha,\beta,\gamma-1)\\
&\quad - q^{p_{12}-\gamma}\,\frac{[\beta]}{\sqrt{[\beta+\gamma]}}\,e_{\boldsymbol{p}}(\alpha+1,\beta-1,\gamma).
\end{aligned} \tag{5.13}$$

It is evident that the coefficients in the right-hand sides of (5.13) are defined unambiguously, since the expressions under the square roots are non-negative, and, for example

$$\lim_{\alpha\to 0}\lim_{\beta\to 0}\frac{[\alpha]^2}{[\alpha+\beta]} = 0 = \lim_{\beta\to 0}\lim_{\alpha\to 0}\frac{[\alpha]^2}{[\alpha+\beta]}.$$

Relations (5.9) and (5.10), on the other hand, have the advantage of admitting a (periodic) continuation to all the integer labels α,β,γ.

In order to find the dimension $d_3(h)$ of each $\mathcal{V}_{\boldsymbol{p}}$, we compute the dimensions of the corresponding weight spaces

$$\mathcal{V}_{\boldsymbol{p}}(\lambda_1,\lambda_2)\ =\ U_q^E(\lambda_1,\lambda_2)\,|\boldsymbol{p};0,0,0\rangle \qquad \dim\,\mathcal{V}_{\boldsymbol{p}}(\lambda_1,\lambda_2)\ =\ \min(\lambda_1,\lambda_2)+1. \tag{5.14}$$

As a result, we find

$$d_3(h) := \dim\,\mathcal{V}_{\boldsymbol{p}} = \sum_{\lambda=0}^{h-1}(\lambda+1)(2(h-\lambda-1)+1) = \frac{h(h+1)(2h+1)}{6}. \tag{5.15}$$

One way of identifying the invariant subspaces of $\mathcal{V}_{\boldsymbol{p}}$ is through the construction of singular vectors. As we shall see shortly, the latter need not belong to the set of PBW basis vectors; in general, they appear as linear combinations of basis vectors of a given weight subspace $U_q^E(\lambda_1,\lambda_2)|\boldsymbol{p};0,0,0\rangle$. The resulting complication is resolved by passing to the canonical basis which does contain the LW (and HW) vectors of interest.

The canonical basis in $\mathcal{V}_{\boldsymbol{p}}$ will be defined by acting on the LW vector $|\boldsymbol{p};0,0,0\rangle$ by two sets of monomials

$$E_1^{(m)}E_2^{(k)}E_1^{(\ell)} \qquad E_2^{(m)}E_1^{(k)}E_2^{(\ell)} \qquad m,k,\ell\in\mathbb{Z}_+ \quad k\geqslant \ell+m$$

which together provide a basis in U_q^+. One proves (see [25]) that the Serre relations (2.3) imply the following expressions for these monomials in terms of the PBW basis

$$\begin{aligned} q^{k\ell} E_1^{(m)} E_2^{(k)} E_1^{(\ell)} &= \sum_{j=0}^{\ell} (-1)^j \binom{m+\ell-j}{m}_+ E_1^{(m+\ell-j)} E_{21}^{[j]} E_2^{(k-j)} \\ q^{k\ell} E_2^{(\ell)} E_1^{(k)} E_2^{(m)} &= \sum_{j=0}^{\ell} (-1)^j \binom{m+\ell-j}{m}_+ E_1^{(k-j)} E_{21}^{[j]} E_2^{(m+\ell-j)}. \end{aligned} \tag{5.16}$$

It follows from (5.16) that

$$E_1^{(m)} E_2^{(k)} E_1^{(\ell)} = E_2^{(\ell)} E_1^{(k)} E_2^{(m)} \qquad \text{for} \quad k = m+\ell. \tag{5.17}$$

The inverse relations are

$$\begin{aligned} E_1^{(\alpha)} E_{21}^{[\beta]} E_2^{(\gamma)} &= \sum_{\sigma=0}^{\beta} X_\sigma(\alpha,\beta,\gamma) E_2^{(\beta-\sigma)} E_1^{(\alpha+\beta)} E_2^{(\gamma+\sigma)} \qquad (\text{for } \alpha \geqslant \gamma) \\ E_1^{(\alpha)} E_{21}^{[\beta]} E_2^{(\gamma)} &= \sum_{\sigma=0}^{\beta} X_\sigma(\gamma,\beta,\alpha) E_1^{(\alpha+\sigma)} E_2^{(\beta+\gamma)} E_1^{(\beta-\sigma)} \qquad (\text{for } \alpha \leqslant \gamma) \end{aligned} \tag{5.18}$$

where

$$X_\sigma(\alpha,\beta,\gamma) = (-1)^{\beta-\sigma} \binom{\gamma+\sigma}{\gamma}_+ q^{\sigma(\sigma-1)+(\beta-\sigma)(\alpha+\beta)}. \tag{5.19}$$

We are now ready to define a canonical basis in $\mathcal{V}_{\boldsymbol{p}}$. It consists of two pieces

$$\begin{aligned} q^{kl} E_1^{(m)} E_2^{(k)} E_1^{(\ell)} \, |\boldsymbol{p}; 0,0,0\rangle \; &:= \; |\boldsymbol{p}; m,k,\ell\rangle^{(1)} \\ q^{kl} E_2^{(\ell)} E_1^{(k)} E_2^{(m)} \, |\boldsymbol{p}; 0,0,0\rangle \; &:= \; |\boldsymbol{p}; \ell,k,m\rangle^{(2)} \end{aligned} \tag{5.20}$$

$(0 \leqslant \ell, m, \; \ell+m \leqslant k \leqslant h-1)$ whose intersection is spanned by

$$|\boldsymbol{p}; m, \ell+m, \ell\rangle^{(1)} \; = \; |\boldsymbol{p}; \ell, \ell+m, m\rangle^{(2)}. \tag{5.21}$$

Using (5.4) and (5.5) we find

$$\begin{aligned} &(q^{H_1} - q^{2m+2\ell-k+1-p_{23}}) \, |\boldsymbol{p}; m,k,\ell\rangle^{(1)} = 0 \\ &(q^{H_1} - q^{2k-m-\ell+1-p_{23}}) \, |\boldsymbol{p}; \ell,k,m\rangle^{(2)} = 0 \\ &(q^{H_2} - q^{2m+2\ell-k+1-p_{12}}) \, |\boldsymbol{p}; \ell,k,m\rangle^{(2)} = 0 \\ &(q^{H_2} - q^{2k-m-\ell+1-p_{12}}) \, |\boldsymbol{p}; m,k,\ell\rangle^{(1)} = 0 \end{aligned} \tag{5.22}$$

$$\begin{aligned} &E_1 \, |\boldsymbol{p}; m,k,\ell\rangle^{(1)} = (m+1)_+ \, |\boldsymbol{p}; m+1,k,\ell\rangle^{(1)} \qquad \text{for} \quad k > m+\ell \\ &E_1 \, |\boldsymbol{p}; \ell,k,m\rangle^{(2)} = (k-\ell+1)_+ |\boldsymbol{p}; \ell,k+1,m\rangle^{(2)} + q^{2(k-\ell+1)}(1-\delta_{\ell 0})(m+1)_+ \\ &\qquad\qquad \times |\boldsymbol{p}; \ell-1,k+1,m+1\rangle^{(2)} \qquad \text{for} \quad k \geqslant m+\ell \\ &E_2 \, |\boldsymbol{p}; m,k,\ell\rangle^{(1)} = q^{m-\ell}(k-m+1)_+ |\boldsymbol{p}; m,k+1,\ell\rangle^{(1)} \\ &\qquad\qquad + \overline{q}^{m+\ell}(1-\delta_{m0})(\ell+1)_+ |\boldsymbol{p}; m-1,k+1,\ell+1\rangle^{(1)} \qquad \text{for} \quad k \geqslant m+\ell \\ &E_2 \, |\boldsymbol{p}; \ell,k,m\rangle^{(2)} = \overline{q}^{k}(\ell+1)_+ |\boldsymbol{p}; \ell+1,k,m\rangle^{(2)} \qquad \text{for} \quad k > m+\ell \end{aligned} \tag{5.23}$$

$$\begin{aligned} &F_1 \, |\boldsymbol{p}; m,k,\ell\rangle^{(1)} = q^{2-p_{23}-k+2\ell}(1-\delta_{m0})(p_{23}-m+k-2\ell)_+ |\boldsymbol{p}; m-1,k,\ell\rangle^{(1)} \\ &\qquad\qquad + q^{2-p_{23}+k}(1-\delta_{\ell 0})(p_{23}-\ell)_+ |\boldsymbol{p}; m,k,\ell-1\rangle^{(1)} \qquad \text{for} \quad k \geqslant m+\ell \\ &F_1 \, |\boldsymbol{p}; \ell,k,m\rangle^{(2)} = q^{2-p_{23}+\ell-m}(p_{23}-k+m)_+ |\boldsymbol{p}; \ell,k-1,m\rangle^{(2)} \qquad \text{for} \quad k > m+\ell \\ &F_2 \, |\boldsymbol{p}; \ell,k,m\rangle^{(2)} = q^{2-p_{12}+2m}(1-\delta_{\ell 0})(p_{12}-\ell+k-2m)_+ |\boldsymbol{p}; \ell-1,k,m\rangle^{(2)} \\ &\qquad\qquad + q^{2-p_{12}}(1-\delta_{m0})(p_{12}-m)_+ |\boldsymbol{p}; \ell,k,m-1\rangle^{(2)} \qquad \text{for} \quad k \geqslant m+\ell \\ &F_2 \, |\boldsymbol{p}; m,k,\ell\rangle^{(1)} = q^{2-p_{12}}(p_{12}-k+\ell)_+ |\boldsymbol{p}; m,k-1,\ell\rangle^{(1)} \qquad \text{for} \quad k > m+\ell. \end{aligned} \tag{5.24}$$

Details of the calculations can be found in the appendix.

The dimension $d_3(h)$ (5.15) can also be recovered by a canonical-basis computation; we have

$$d_3(h) = 2\sum_{\lambda_1=0}^{h-1}\sum_{\lambda_2=0}^{\lambda_1-1}(\lambda_2+1) + \sum_{\lambda=0}^{h-1}(\lambda+1) = \sum_{\ell=1}^{h}\ell^2 = \frac{h(h+1)(2h+1)}{6}. \quad (5.25)$$

Remark 5.1. Consider the algebra $U^E_{(h)}$ obtained from U^E_q by factoring the latter with respect to the relations $E_a^h = 0 = [hH]$ and adding the new elements $E_a^{(h)}$ (of [25]) such that

$$[q^{H_a}, E_a^{(h)}] = 0 \qquad [E_a, E_a^{(h)}] = 0 \qquad [q^{H_{3-a}}, E_a^{(h)}]_+ = 0 \qquad a = 1, 2. \quad (5.26)$$

We can define a periodic extension of $\mathcal{V}_p$ if we impose the relations

$$[E_{3-a}, E_a^{(h)}]_+\mathcal{V}_p = 0 \qquad (E_2^{(h)}E_1^{(h)} - (-1)^h E_1^{(h)}E_2^{(h)})\mathcal{V}_p = 0 \qquad [F_a, E_b^{(h)}]\mathcal{V}_p = 0 \quad (5.27)$$

and the periodicity condition

$$\left(\left(E_a^{(h)}\right)^2 - 1\right)\mathcal{V}_p = 0. \quad (5.28)$$

(In view of (4.6), $(E_a^{(h)})^2 = 2E_a^{(2h)}$.) Indeed, there is an ideal $\mathcal{J}_{(h)}$ of $U^E_{(h)}$ generated by

$$[E_1, E_2^{(h)}]_+ \qquad [E_2, E_1^{(h)}]_+ \qquad E_2^{(h)}E_1^{(h)} - (-1)^h E_1^{(h)}E_2^{(h)}.$$

Then equations (5.18) and (5.19) imply that, in $U^E_{(h)}/\mathcal{J}_{(h)}$

$$E_1^{(h-\beta)}E_{21}^{[\beta]} = \sum_{\sigma=0}^{\beta} q^{\sigma(\sigma-1)}E_2^{(\beta-\sigma)}E_1^{(h)}E_2^{(\sigma)} = \sum_{\sigma=0}^{\beta}(-1)^\sigma q^{\sigma(\sigma-1)}\binom{\beta}{\sigma}_+ E_2^{(\beta)}E_1^{(h)} = 0 \quad (5.29)$$

for any $\beta > 0$, where we are using the relation $E_1^{(h)}E_2^{(\sigma)} = (-1)^\sigma E_2^{(\sigma)}E_1^{(h)}$ and the identity

$$\sum_{\sigma=0}^{\beta}(-1)^\sigma q^{\sigma(\sigma-1)}\binom{\beta}{\sigma}_+ = \delta_{\beta 0} \qquad \text{for} \quad \beta \in \mathbb{Z}_+ \quad (5.30)$$

(essentially, a q-version of $(1-1)^\beta = 0$ for $\beta > 0$). It follows, in particular, that

$$E_{21}^{[h]}|\boldsymbol{p}; 0, 0, 0\rangle = 0. \quad (5.31)$$

Remark 5.2. The space $\mathcal{V}_p$ can also be defined in terms of the PBW basis for $U_q(s\ell_n)$ with $n > 3$. The basis involves $\binom{n}{2}$ exponents $\alpha_{ij} = \alpha_{ji} \in \mathbb{Z}_+$ satisfying $\sum_j \alpha_{ij} \leqslant h - 1$. The dimension $d_n(h)$ of $\mathcal{V}_p$ is a polynomial in h of degree $\binom{n}{2}$; for $n = 4$ it is given by

$$d_4(h) = \frac{1}{6!}\,h(h+1)^2(h+2)(h+3)(11h+4). \quad (5.32)$$

5.2. BRS intertwiners, singular vectors and invariant subspaces

The presence of two (equivalent) bases in $\mathcal{V}_p$ is an asset: it enables us to use for each problem the one which is better adapted to its solution. We shall illustrate this fact by writing down the BRS intertwiners

$$Q_p : \mathcal{V}_{w_{\mathrm{L}}p} \to \mathcal{V}_p \qquad Q_{h+w_{\mathrm{L}}p} : \mathcal{V}_p \to \mathcal{V}_{h+w_{\mathrm{L}}p} \quad (5.33)$$

in the PBW basis and the singular vectors in $\mathcal{V}_p$ in the canonical basis.

Let us start with Q_p. From q^{H_i} invariance (5.7) it follows that

$$Q_p|w_{\mathrm{L}}\boldsymbol{p}; \alpha, \beta, \gamma\rangle = \sum_\rho f_\rho(\alpha, \beta, \gamma; \boldsymbol{p})|\boldsymbol{p}; \alpha+\rho, \beta-\rho+p_{13}, \gamma+\rho\rangle. \quad (5.34)$$

The E_1 and E_{21} invariance (cf (5.9)) implies

$$f_\rho(\alpha,\beta,\gamma;\boldsymbol{p}) = q^{(\alpha+\beta)\rho}\begin{bmatrix}\beta+p_{13}-\rho\\ \beta\end{bmatrix}\begin{bmatrix}\alpha+\rho\\ \alpha\end{bmatrix} f_\rho(0,0,\gamma;\boldsymbol{p}). \tag{5.35}$$

To find $f_\rho(0,0,\gamma;\boldsymbol{p}) \equiv f_\rho(\gamma;\boldsymbol{p})$, it is convenient to use the equations following from E_2 and F_1 invariance since they contain the same triples of various ρ and γ combinations. Applying E_2, one gets

$$\begin{aligned} q^{\beta}\,(\gamma+1)_+\, f_\rho(\alpha,\beta,\gamma+1;\boldsymbol{p}) - q^{-\beta}\,(\beta+1)_+\, f_{\rho+1}(\alpha-1,\beta+1,\gamma;\boldsymbol{p}) \\ = q^{\beta-2\rho+p_{13}}(\gamma+\rho+1)_+ f_\rho(\alpha,\beta,\gamma;\boldsymbol{p}) \\ -\,q^{-\beta-p_{13}}\,(\beta-\rho+p_{13})_+\, f_{\rho+1}\,(\alpha,\beta,\gamma;\boldsymbol{p}) \end{aligned} \tag{5.36}$$

whereas F_1 invariance implies

$$\begin{aligned} q^{-\alpha-\beta-p_{12}+1}\,[\gamma-\alpha-\beta-p_{12}]\, f_{\rho+1}(\alpha-1,\beta,\gamma;\boldsymbol{p}) \\ = q^{-\rho-\alpha-\beta-p_{12}}\,[\gamma+\rho-\alpha-\beta-p_{12}+1]\, f_{\rho+1}(\alpha,\beta,\gamma;\boldsymbol{p}) \\ +\,q^{-\rho}\,[\gamma+\rho+1]\, f_\rho(\alpha,\beta,\gamma;\boldsymbol{p}) - [\gamma+1]\, f_\rho(\alpha,\beta-1,\gamma+1;\boldsymbol{p}). \end{aligned} \tag{5.37}$$

After some algebra one obtains the following simple and selfconsistent (e.g. not containing α and β) recurrence relations for the function $f_\rho(\gamma;\boldsymbol{p})$:

$$\begin{aligned} f_{\rho+1}(\gamma;\boldsymbol{p}) &= -\frac{(\gamma+\rho+1)_+}{(\gamma+\rho-p_{12}+1)_+}\, f_\rho(\gamma;\boldsymbol{p}) \\ f_\rho(\gamma+1;\boldsymbol{p}) &= \frac{[\gamma+\rho+1][\gamma+p_{23}+1]}{[\gamma+1][\gamma+\rho-p_{12}+1]}\, f_\rho(\gamma;\boldsymbol{p}). \end{aligned} \tag{5.38}$$

Solving the recurrence relations (5.38) for $f_\rho(\gamma;\boldsymbol{p})$ and putting the latter in (5.35), one can bring the final result for $f_\rho(\alpha,\beta,\gamma;\boldsymbol{p})$ to the form

$$\begin{aligned} f_\rho(\alpha,\beta,\gamma;\boldsymbol{p}) &= (-q^{(\alpha+\beta+p_{12})})^{\rho}\begin{bmatrix}\beta+p_{13}-\rho\\ \beta\end{bmatrix}\begin{bmatrix}\alpha+\rho\\ \alpha\end{bmatrix}\begin{bmatrix}\gamma+\rho\\ p_{12}\end{bmatrix}\begin{bmatrix}\gamma+p_{23}\\ p_{23}\end{bmatrix} f(\boldsymbol{p}) \\ &= (-q^{(\alpha+\beta+p_{12})})^{\rho}\,\frac{[\alpha+\rho]![\beta+p_{13}-\rho]![\gamma+\rho]![\gamma+p_{23}]!}{[\rho]![\alpha]![\beta]![\gamma]![p_{13}-\rho]![\gamma-p_{12}+\rho]!}\,\frac{f(\boldsymbol{p})}{[p_{12}]![p_{23}]!}. \end{aligned} \tag{5.39}$$

One sees that the effective summation range over ρ in (5.34) (for $0 < p_{13} < h$) is from $\max\,(p_{12}-\gamma,0)$ to $\min\,(p_{13},h-\alpha,h-\gamma)$.

One can find quite analogously the corresponding expression for the action $Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}}$. It turns out that

$$\begin{aligned} Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}}|\boldsymbol{p};\alpha,\beta,\gamma\rangle = \sum_\rho f_\rho(\alpha,\beta,\gamma;h-p_{23},h-p_{12}) \\ \times|w_{\rm L}\boldsymbol{p};\alpha+\rho,\beta+2h-p_{13}-\rho,\gamma+\rho\rangle. \end{aligned} \tag{5.40}$$

It is easy to see that successive application of $Q_{\boldsymbol{p}}$ and $Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}}$ gives zero, i.e.

$$Q_{\boldsymbol{p}}Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}} = 0 = Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}}Q_{\boldsymbol{p}}. \tag{5.41}$$

This follows from the explicit form of the coefficient functions $f_\rho(\alpha,\beta,\gamma;\boldsymbol{p})$. Indeed (for $f(\boldsymbol{p})=1$)

$$\begin{aligned} Q_{\boldsymbol{h}+w_{\rm L}\boldsymbol{p}}Q_{\boldsymbol{p}}|\boldsymbol{h}+w_{\rm L}\boldsymbol{p};\alpha,\beta,\gamma\rangle = \sum_\sigma |w_{\rm L}\boldsymbol{p};\alpha+\sigma,\beta+2h-\sigma,\gamma+\sigma\rangle \\ \times\sum_\rho f_\rho(\alpha,\beta,\gamma;\boldsymbol{p}) f_{\sigma-\rho}(\alpha+\rho,\beta+p_{13}-\rho,\gamma+\rho;h-p_{23},h-p_{12}) \end{aligned} \tag{5.42}$$

and

$$f_\rho(\alpha,\beta,\gamma;\boldsymbol{p})\, f_{\sigma-\rho}(\alpha+\rho,\beta+p_{13}-\rho,\gamma+\rho;h-p_{23},h-p_{12})$$
$$= \frac{[\alpha+\sigma]![\beta+2h-\sigma]![\gamma+\sigma]![\gamma+\rho+h-p_{12}]!}{[\rho]![\alpha]![\beta]![\gamma]![\gamma+\rho-p_{12}]![\sigma-\rho]![p_{12}]![p_{23}]![h-p_{12}]![h-p_{23}]!}$$
$$\times \frac{[\gamma+p_{23}]!}{[p_{13}-\rho]![2h-p_{13}+\rho-\sigma]![\gamma+\sigma-h+p_{23}]!} \equiv 0 \qquad (5.43)$$

(due to factorials of integers greater or equal to h in the numerator, or factorials of negative integers in the denominator).

For similar reasons

$$Q_{\boldsymbol{p}} Q_{\boldsymbol{h}+w_{\mathrm{L}}\boldsymbol{p}}|\boldsymbol{p};\alpha,\beta+p_{13},\gamma\rangle = \sum_\sigma |\boldsymbol{h}+\boldsymbol{p};\alpha+\sigma,\beta+p_{13}-\sigma,\gamma+\sigma\rangle$$
$$\times \sum_\mu f_\mu(\alpha,\beta+p_{13},\gamma;h-p_{23},h-p_{12}) f_{\sigma-\mu}(\alpha+\mu,\beta+2h-\mu,\gamma+\mu;\boldsymbol{p}) \qquad (5.44)$$

also vanishes identically.

One should expect the vector $Q_{\boldsymbol{p}}|w_{\mathrm{L}}\boldsymbol{p};0,0,0\rangle \in \mathcal{V}_{\boldsymbol{p}}$ to be *singular*. In view of (5.34) and (5.39), it can be shown to be proportional to

$$|s_{\boldsymbol{p}}\rangle = \sum_{\alpha=p_{12}}^{p_{13}} (-1)^\alpha \begin{pmatrix} \alpha \\ p_{12} \end{pmatrix}_+ |\boldsymbol{p};\alpha,p_{13}-\alpha,\alpha\rangle. \qquad (5.45)$$

The dual formulae are

$$|s_{\boldsymbol{h}+w_{\mathrm{L}}\boldsymbol{p}}\rangle = \sum_{\alpha=h-p_{23}}^{2h-p_{13}} (-1)^\alpha \begin{pmatrix} \alpha \\ h-p_{23} \end{pmatrix}_+ |\boldsymbol{h}+w_{\mathrm{L}}\boldsymbol{p};\alpha,2h-p_{13}-\alpha,\alpha\rangle \sim Q_{\boldsymbol{h}+w_{\mathrm{L}}\boldsymbol{p}}|\boldsymbol{p};0,0,0\rangle. \qquad (5.46)$$

Applying (5.10), one can check that the weight vectors (5.45) and (5.46) indeed satisfy

$$F_a\,|s_{\boldsymbol{p}}\rangle = 0 = F_a\,|s_{\boldsymbol{h}+w_{\mathrm{L}}\boldsymbol{p}}\rangle \qquad a=1,2. \qquad (5.47)$$

The expressions for the singular vectors become particularly elegant in the canonical basis (a similar remark was made in [21]). Noting that for $m+\ell=k$ equations (5.16) reduce to

$$q^{k\ell}E_1^{(k-\ell)}E_2^{(k)}E_1^{(\ell)} \equiv q^{k\ell}E_2^{(\ell)}E_1^{(k)}E_2^{(k-\ell)} = \sum_{j=0}^{\ell}(-1)^j \begin{pmatrix} k-j \\ m \end{pmatrix}_+ E_1^{(k-j)}E_{21}^{[j]}E_2^{(k-j)} \qquad (5.48)$$

and substituting $m=p_{12},\ \ell=p_{23},\ k=p_{13},\ j=p_{13}-\alpha$, one obtains

$$q^{p_{13}p_{23}}\,E_1^{(p_{12})}E_2^{(p_{13})}E_1^{(p_{23})} \equiv q^{p_{13}p_{23}}\,E_2^{(p_{23})}E_1^{(p_{13})}E_2^{(p_{12})}$$
$$= (-1)^{p_{13}} \sum_{\alpha=p_{12}}^{p_{13}} (-1)^\alpha \begin{pmatrix} \alpha \\ p_{12} \end{pmatrix}_+ E_1^{(\alpha)}E_{21}^{[p_{13}-\alpha]}E_2^{(\alpha)}. \qquad (5.49)$$

Hence, for $|s_{\boldsymbol{p}}\rangle \in \mathcal{V}_{\boldsymbol{p}}$ we have

$$|\boldsymbol{p};p_{12},p_{13},p_{23}\rangle^{(1)} \equiv |\boldsymbol{p};p_{23},p_{13},p_{12}\rangle^{(2)} = (-1)^{p_{13}}|s_{\boldsymbol{p}}\rangle. \qquad (5.50)$$

The identification of invariant subspaces and quotients of $\mathcal{V}_{\boldsymbol{p}}$ is facilitated by the knowledge of the invariant Hermitean form (which majorizes the invariant bilinear form; cf remark 3.4). Noting that $(E_a^{(m)})^* = \frac{1}{(m)_-!}F_a^m$ and observing the identity $(m)_+(m)_- = [m]^2$, we can write the following expression for the norm square of canonical-basis vectors:

$$\||\boldsymbol{p};m,k,\ell\rangle^{(1)}\|^2 = \langle\boldsymbol{p};0,0,0|F_1^{[\ell]}\,F_2^{[k]}\,F_1^{[m]}\,E_1^{[m]}\,E_2^{[k]}\,E_1^{[\ell]}\,|\boldsymbol{p};0,0,0\rangle \quad (5.51)$$

and a similar expression for $1 \leftrightarrow 2$ and $\ell \leftrightarrow m$ which is completely determined if we set $\langle\boldsymbol{p};0,0,0|\boldsymbol{p};0,0,0\rangle = 1$ (for $1 < p_{13} < h$).

Remark 5.3. We can surely also compute the invariant bilinear and Hermitean forms in the PBW basis. For instance, one can derive the relation

$$\langle \boldsymbol{p}; \alpha, \beta, \gamma | \boldsymbol{p}; \alpha, \beta, \gamma \rangle = \sum_{\sigma=\max(\alpha-\beta,0)}^{\alpha} (-1)^{\alpha+\sigma} \begin{bmatrix} \alpha+\gamma-\sigma \\ \gamma \end{bmatrix}^2 \times \begin{bmatrix} p_{23}+\gamma-\beta-1 \\ \sigma \end{bmatrix} \begin{bmatrix} p_{13}+\sigma-\alpha-\gamma-2 \\ \beta-\alpha+\sigma \end{bmatrix} \begin{bmatrix} p_{12}-1 \\ \alpha+\gamma-\sigma \end{bmatrix}. \tag{5.52}$$

Here the advantage of the canonical basis becomes manifest: for a singular vector of the type (5.50) the vanishing of its norm square (5.51) is an immediate consequence of (5.47). By contrast, the (bilinear) squares (5.52) of PBW basis vectors entering the expansion (5.45) are not, in general, zero; only the resulting sum of inner products should vanish.

A calculation similar to the (canonical basis) computation of $d_3(h)$ yields the dimension of the image of $Q_{\boldsymbol{p}}$ in $\mathcal{V}_{\boldsymbol{p}}$

$$\dim\,(\mathrm{Im}\ Q_{\boldsymbol{p}}) = \dim\,(\mathrm{Ker}\ Q_{\boldsymbol{h}+w_{\mathrm{L}}\boldsymbol{p}}) = \tfrac{1}{6}(h-p_{13})(h-p_{13}+1)(2h-2p_{13}+1). \tag{5.53}$$

This invariant subspace lies in the kernel of the Hermitean form on $\mathcal{V}_{\boldsymbol{p}}$ but does not exhaust it.

A systematic study of the structure of invariant subspaces and subquotients of $\mathcal{V}_{\boldsymbol{p}}$ is left for future work.

6. Concluding remarks

The preceding discussion being rather technical in nature, it should be helpful to sum up the main results adding on the way some new emphases.

We are studying indecomposable representations of $U_q(s\ell_n)$ which are trivial on the ideal generated by

$$E_a^h \quad F_a^h \quad [hH_a] \qquad a=1,\ldots,n-1 \quad h\in\mathbb{N} \quad h>n \tag{6.1}$$

for q a $2h$th root of unity. The resulting finite-dimensional factor algebra has a surprisingly rich structure of indecomposable representations that parallel the structure of indecomposable Kac–Moody modules. Different dual pairs of h-dimensional $U_q(s\ell_2)$ representations of this kind are studied in section 3. The most economic construction deals with a pair of indecomposable modules $\mathcal{V}_p$ and $\mathcal{V}_{2h-p}$ which already exhibit the counterpart of the Bernard–Felder cohomology we are interested in. The rather complete analysis of the structure of the h-dimensional representations of $U_q(s\ell_2)$ in $\mathcal{V}_p$ and of the intertwining ('BRS') maps $Q^{h-p} : \mathcal{V}_p \to \mathcal{V}_{2h-p}$, performed in section 3, allowed us to reconstruct (in section 4) the $2h$-dimensional indecomposable modules $\mathcal{W}_p$ studied earlier in [15–18].

It is remarkable that the $U_q(s\ell_n)$ counterparts $(\mathcal{V}_{\boldsymbol{p}}, \mathcal{V}_{\boldsymbol{h}+w_{\mathrm{L}}\boldsymbol{p}})$ of the above dual pairs can be constructed effectively for any n using the PBW basis. This is demonstrated by an explicit calculation in section 5 for $n=3$. It is shown that Lusztig's canonical basis (which in this case also consists of monomials in the raising operators) provides better control over singular vectors. A detailed study of the structure of invariant subspaces and subquotients of $\mathcal{V}_p$, however, has only been performed for $n=2$.

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Appendix

Here we give the details, for $n = 3$, of the most involved calculation in the transition from PBW to the canonical basis that contains all the needed technical steps. It is assumed that $k \geqslant m + \ell$.

$$
\begin{aligned}
E_2\, E_1^{(m)} E_2^{(k)} E_1^{(\ell)} |\boldsymbol{p}; 0, 0, 0\rangle &= \bar{q}^{k\ell} \sum_{j=0}^{\ell} (-1)^j \binom{m+\ell-j}{m}_+ E_2 |\boldsymbol{p}; m+\ell-j, j, k-j\rangle \\
&= \bar{q}^{m+\ell(k+1)} \sum_{j=0}^{\ell} (-1)^j \binom{m+\ell-j}{m}_+ \\
&\quad \times \{ q^{2j} (k-j+1)_+ |\boldsymbol{p}; m+\ell-j, j, k-j+1\rangle \\
&\quad - (j+1)_+ |\boldsymbol{p}; m+\ell-j-1, j+1, k-j\rangle \} \\
&= \bar{q}^{m+\ell(k+1)} \sum_{j=0}^{\ell+1} (-1)^j |\boldsymbol{p}; m+\ell-j, j, k-j+1\rangle \\
&\quad \times \left\{ \binom{m+\ell-j}{m}_+ q^{2j} (k-j+1)_+ + \binom{m+\ell+1-j}{m}_+ (j)_+ \right\} \\
&= \bar{q}^{m+\ell(k+1)} \sum_{j=0}^{\ell+1} (-1)^j \\
&\quad \times \left\{ \binom{m+\ell-j}{m}_+ q^{2j} (k-j+1)_+ + \binom{m+\ell+1-j}{m}_+ (j)_+ \right\} \\
&\quad \times \sum_{n=0}^{j} (-1)^{j-n} \binom{m+\ell-j+n}{n}_+ \\
&\quad \times q^{n(n-1)+(j-n)(k+1)} E_1^{(m+\ell-j+n)} E_2^{(k+1)} E_1^{(j-n)} |\boldsymbol{p}; 0, 0, 0\rangle
\end{aligned}
$$

(now we change the summation index n by $j - n$)

$$
\begin{aligned}
&= \bar{q}^{m+\ell(k+1)} \sum_{j=0}^{\ell+1} (-1)^j \\
&\quad \times \left\{ \binom{m+\ell-j}{m}_+ q^{2j} (k-j+1)_+ + \binom{m+\ell+1-j}{m}_+ (j)_+ \right\} \\
&\quad \times \sum_{n=0}^{j} (-1)^n \binom{m+\ell-n}{j-n}_+ q^{(j-n)(j-n-1)+n(k+1)} \\
&\quad \times E_1^{(m+\ell-n)} E_2^{(k+1)} E_1^{(n)} |\boldsymbol{p}; 0, 0, 0\rangle
\end{aligned}
$$

(here we can change the upper summation limit in the sum over n from j to $\ell + 1$ since the binomial coefficient $\binom{m+\ell-n}{j-n}_+$ is automatically zero when $n \geqslant j$; after that we can exchange the orders of summation over n and j)

$$= \bar{q}^{m+\ell(k+1)} \sum_{n=0}^{\ell+1} (-1)^n q^{n(k+1)} \sum_{j=0}^{\ell+1} (-1)^j \binom{m+\ell-n}{j-n}_+ q^{(j-n)(j-n-1)}$$
$$\times \left\{ \binom{m+\ell-j}{m}_+ q^{2j}(k-j+1)_+ + \binom{m+\ell+1-j}{m}_+ (j)_+ \right\}$$
$$\times E_1^{(m+\ell-n)} E_2^{(k+1)} E_1^{(n)} |\boldsymbol{p}; 0, 0, 0\rangle. \tag{A.1}$$

Let us now compute the sum over j:

$$\sum_{j=0}^{\ell+1} (-1)^j \binom{m+\ell-n}{j-n}_+ q^{(j-n)(j-n-1)}$$
$$\times \left\{ \binom{m+\ell-j}{m}_+ q^{2j}(k-j+1)_+ + \binom{m+\ell+1-j}{m}_+ (j)_+ \right\}$$
$$= \sum_{j=0}^{\ell+1} (-1)^j q^{(j-n)(j-n-1)} \frac{(m+\ell-n)_+!}{(m)_+!(j-n)_+!(\ell+1-j)_+!}$$
$$\times \left\{ q^{2j}(k+1-j)_+(\ell+1-j)_+ + (j)_+(m+\ell+1-j)_+ \right\}$$

(crucial observation:

$$q^{2j}(k+1-j)_+(\ell+1-j)_+ + (j)_+(m+\ell+1-j)_+$$
$$= (m+\ell+1-j)_+(k-m+1)_+ + q^{2(k-m+1)}(m)_+(\ell-k+m)_+$$

the second expression being 'q-linear' in j)

$$= \frac{(m+\ell-n)_+!}{(m)_+!(\ell+1-n)_+!} \sum_{j=0}^{\ell+1} (-1)^j q^{(j-n)(j-n-1)} \binom{\ell+1-n}{j-n}_+$$
$$\times \left\{ (m+\ell+1-j)_+(k-m+1)_+ + q^{2(k-m+1)}(m)_+(\ell-k+m)_+ \right\}$$

(denoting $j-n$ by j)

$$= \frac{(m+\ell-n)_+!}{(m)_+!(\ell+1-n)_+!} (-1)^n \sum_{j=0}^{\ell+1-n} (-1)^j q^{j(j-1)} \binom{\ell+1-n}{j}_+$$
$$\times \left\{ (m+\ell-n+1-j)_+(k-m+1)_+ + q^{2(k-m+1)}(m)_+(\ell-k+m)_+ \right\}$$

(using

$$\sum_{j=0}^{n} (-1)^j q^{j(j-1)} \binom{n}{j}_+ (M-j)_+ = (M)_+\delta_{n0} + q^{2(M-1)}\delta_{n1} \tag{A.2}$$

and (5.30) with $\beta \to n$, $\sigma \to j$)

$$= \frac{(m+\ell-n)_+!}{(m)_+!(\ell+1-n)_+!} (-1)^n \{ q^{2m}(k-m+1)_+\delta_{\ell n}$$
$$+ (m)_+\left[(k-m+1)_+ + q^{2(k-m+1)}(\ell-k+m)_+\right]\delta_{\ell+1\, n} \}$$
$$= \frac{(m+\ell-n)_+!}{(m)_+!(\ell+1-n)_+!} (-1)^n \left\{ q^{2m}(k-m+1)_+\delta_{\ell n} + (m)_+(\ell+1)_+\delta_{\ell+1\, n} \right\}$$
$$= (-1)^n \left\{ q^{2m}(k-m+1)_+\delta_{\ell n} + (\ell+1)_+\delta_{\ell+1\, n} \right\}. \tag{A.3}$$

Combining (A.1) and (A.3), we get eventually (for $k \geqslant m + \ell$)

$$
\begin{aligned}
E_2 E_1^{(m)} E_2^{(k)} E_1^{(\ell)} |\boldsymbol{p}; 0, 0, 0\rangle &= \sum_{n=0}^{\ell+1} \bar{q}^{(\ell-n)(k+1)+m} \\
&\quad \times \left\{ q^{2m} (k-m+1)_+ \delta_{\ell n} + (\ell+1)_+ \delta_{\ell+1\, n} \right\} E_1^{(m+\ell-n)} E_2^{(k+1)} E_1^{(n)} |\boldsymbol{p}; 0, 0, 0\rangle \\
&= q^m (k-m+1)_+ E_1^{(m)} E_2^{(k+1)} E_1^{(\ell)} |\boldsymbol{p}; 0, 0, 0\rangle \\
&\quad + q^{k-m+1} (\ell+1)_+ E_1^{(m-1)} E_2^{(k+1)} E_1^{(\ell+1)} |\boldsymbol{p}; 0, 0, 0\rangle. \qquad \text{(A.4)}
\end{aligned}
$$

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